IEOR 265 – Lecture 14
(Robust) Linear Tube MPC

1 LTI System with Uncertainty

Suppose we have an LTI system in discrete time with disturbance:

$$x_{n+1} = Ax_n + Bu_n + d_n,$$

where $d_n \in \mathcal{W}$ for a bounded polytope $\mathcal{W}$. This disturbance can represent, for instance, modeling error. As an example, suppose that the true system model is given by

$$x_{n+1} = Ax_n + Bu_n + g(x_n, u_n),$$

where $g(\cdot, \cdot)$ is a nonlinear function that obeys $g(x_n, u_n) \in \mathcal{W}$ for all $x_n \in \mathcal{X}$ and $u_n \in \mathcal{U}$. Note that this is not restrictive because it can describe a general nonlinear system

$$x_{n+1} = f(x_n, u_n)$$

by choosing $g(x_n, u_n) = f(x_n, u_n) - Ax_n - Bu_n$. An important question for the LTI system with uncertainty is whether a constant state-feedback controller $u_n = Kx_n$ ensures that all state and input constraints are satisfied for all time.

2 Polytope Operations

To be able to analyze and do design, we need to define operations on polytopes. Let $\mathcal{U}, \mathcal{V}, \mathcal{W}$ by polytopes. The *Minkowski sum* is $\mathcal{U} \oplus \mathcal{V} = \{u + v : u \in \mathcal{U}; v \in \mathcal{V}\}$, and the *Pontryagin set difference* is $\mathcal{U} \ominus \mathcal{V} = \{u : u \oplus \mathcal{V} \subseteq \mathcal{U}\}$. Note that this set difference is not symmetric, meaning that in general $\mathcal{U} \ominus \mathcal{V} \neq \mathcal{V} \ominus \mathcal{U}$; however, the Minkowski sum can be seen to be symmetric by its definition. Some useful properties include

- $(\mathcal{U} \ominus \mathcal{V}) \oplus \mathcal{V} \subseteq \mathcal{U}$;
- $(\mathcal{U} \ominus (\mathcal{V} \oplus \mathcal{W})) \oplus \mathcal{W} \subseteq \mathcal{U} \ominus \mathcal{V}$;
- $(\mathcal{U} \ominus \mathcal{V}) \ominus \mathcal{W} \subseteq \mathcal{U} \ominus (\mathcal{V} \oplus \mathcal{W})$;
- $T(\mathcal{U} \ominus \mathcal{V}) \subseteq T\mathcal{U} \ominus T\mathcal{V}$, where T is a linear transformation.

3 Maximum Output Admissible Disturbance Invariant Set

We return to the question: Does there exist a set Ω such that if $x_0 \in \Omega$, then the controller $u_n = Kx_n$ ensures that both state and input constraints are satised for the LTI system with bounded disturbance. In mathematical terms, we would like this set Ω to achieve (a) constraint satisfaction

$$\Omega \subseteq \{x : x \in \mathcal{X}; Kx \in \mathcal{U}\},$$

and (b) disturbance invariance

$$(A + BK)\Omega \oplus \mathcal{W} \subseteq \Omega.$$

It turns out that we can compute such a set under reasonable conditions on the disturbance $\mathcal{W}$ and constraint sets $\mathcal{X}$ and $\mathcal{U}$. However, there is no guarantee that this is not the null set: It may be the case that $\Omega = \emptyset$. The algorithm for computing this set is more complicated than the situation without disturbance, and so we will not discuss it in this class.

4 Optimization Formulation

Recall the LTI system with disturbance:

$$x_{n+1} = Ax_n + Bu_n + d_n,$$

where $d_n \in \mathcal{W}$. The issue with using a constant state-feedback controller $u_n = Kx_n$ is that the set Ω for which this controller is admissible and ensures constraint satisfaction subject to all possible sequences of disturbances (i.e., $d_0, d_1, \ldots$) can be quite small. In this sense, using this controller is conservative, and the relevant question is whether we can design a nonlinear controller $u_n = \ell(x_n)$ for the LTI system with disturbance such that constraints on states $x_n \in \mathcal{X}$ and inputs $u_n \in \mathcal{U}$ are satisfied. One idea is to use an MPC framework with robustness explicitly included.

(Robust) linear tube MPC is one approach for this. The idea underlying this approach is to subtract out the effect of the disturbance from our state and input constraints. One formulation is

$$\begin{aligned} V_n(x_n) = \min \psi_n(\overline{x}_n, \ldots, \overline{x}_{n+N}, \check{u}_n, \ldots, \check{u}_{n+N-1}) \\ \text{s.t. } & \overline{x}_{n+1+k} = A\overline{x}_{n+k} + B\check{u}_{n+k}, \forall k = 0, \ldots, N-1 \\ & \overline{x}_n = x_n \\ & \overline{x}_{n+k} \in \mathcal{X} \ominus \mathcal{R}_k, \forall k = 1, \ldots, N-1 \\ & \check{u}_{n+k} = K\overline{x}_{n+k} + c_k, \forall k = 0, \ldots, N-1 \\ & \check{u}_{n+k} \in \mathcal{U} \ominus K\mathcal{R}_k, \forall k = 0, \ldots, N-1 \\ & \overline{x}_{n+N} \in \Omega \ominus \mathcal{R}_N \end{aligned}$$

where $\mathcal{X}, \mathcal{U}, \Omega$ are polytopes, $\mathcal{R}_0 = \{0\}$, $\mathcal{R}_k = \oplus_{j=0}^{k-1}(A + BK)^j \mathcal{W}$ is a disturbance tube, and $N > 0$ is the horizon. Note that we do not constrain the initial condition x_n, and this reduces the conservativeness of the MPC formulation. We will refer to the set Ω as a terminal set. The interpretation of the function ψ_n is a cost function on the states and inputs of the system.

5 Recursive Properties

Suppose that (A, B) is stabilizable and K is a matrix such that $(A + BK)$ is stable. For this given system and feedback controller, suppose we have a maximal output admissible disturbance invariant set Ω (meaning that this set has constraint satisfaction $\Omega \subseteq \{x : x \in \mathcal{X}, Kx \in \mathcal{U}\}$ and disturbance invariance $(A + BK)\Omega \oplus \mathcal{W} \subseteq \Omega$).
Next, note that conceptually our decision variables are c_k since the $\overline{x}_k, \check{u}_k$ are then uniquely determined by the initial condition and equality constraints. As a result, we will talk about solutions only in terms of the variables c_k. In particular, if $\mathcal{M}_n = \{c_n, \ldots, c_{n+N-1}\}$ is feasible for the optimization defining linear MPC with an initial condition x_n, then the system that applies the control value $u_n = Kx_n + c_n[\mathcal{M}_n]$ results in:

- Recursive Feasibility: there exists feasible $\mathcal{M}_{n+1}$ for x_{n+1};
- Recursive Constraint Satisfaction: $x_{n+1} \in \mathcal{X}$.

We will give a sketch of the proof for these two results.

- Choose $\mathcal{M}_{n+1} = \{c_1[\mathcal{M}_n], \ldots, c_{N-1}[\mathcal{M}_n], 0\}$. Some algebra gives that $\overline{x}_{n+1+k}[\mathcal{M}_{n+1}] = \overline{x}_{n+1+k}[\mathcal{M}_n] + (A+BK)^k d_n \in \overline{x}_{n+1+k}[\mathcal{M}_n] \oplus (A+BK)^k\mathcal{W}$. But because $\mathcal{M}_n$ is feasible, it must be that $\overline{x}_{n+1+k}[\mathcal{M}_n] \in \mathcal{X} \ominus \mathcal{R}_{k+1}$. Combining this gives

  $$\begin{aligned} \overline{x}_{n+1+k}[\mathcal{M}_{n+1}] \in \overline{x}_{n+1+k}[\mathcal{M}_n] \oplus (A + BK)^k\mathcal{W} & \\ \subseteq \mathcal{X} \ominus (\mathcal{R}_k \oplus (A + BK)^k\mathcal{W}) \oplus (A + BK)^k\mathcal{W} & \\ \subseteq \mathcal{X} \ominus \mathcal{R}_k. & \end{aligned}$$

  This shows feasibility for the states; a similar argument can be applied to the inputs.

  The final state needs to be considered separately. The argument above gives

  $$\overline{x}_{n+1+N-1}[\mathcal{M}_{n+1}] \in \Omega \ominus \mathcal{R}_{N-1}.$$

  Observe that the constraint satisfaction property of Ω leads to

  $$\check{u}_{n+1+N-1}[\mathcal{M}_{n+1}] = K\overline{x}_{n+1+N-1}[\mathcal{M}_{n+1}] \subseteq K\Omega \ominus K\mathcal{R}_{N-1} \subseteq \mathcal{U} \ominus K\mathcal{R}_{N-1}.$$

  Lastly, note that the disturbance invariance property of Ω gives

  $$\begin{aligned} \overline{x}_{n+1+N}[\mathcal{M}_{n+1}] \in &(A + BK)\Omega \ominus (A + BK)\mathcal{R}_{N-1} \\ &\subseteq (\Omega \ominus \mathcal{W}) \ominus (A + BK)\mathcal{R}_{N-1} = \Omega \ominus \mathcal{R}_N. \end{aligned}$$

- Since $x_{n+1} = \overline{x}_{n+1}[\mathcal{M}_n] + d_n \in (\mathcal{X} \ominus \mathcal{W}) \oplus \mathcal{W} \subseteq \mathcal{X}$, the result follows.

There is an immediate corollary of this theorem: If there exists feasible $\mathcal{M}_0$ for x_0, then the system is (a) Lyapunov stable, (b) satisfies all state and input constraints for all time, (c) feasible for all time.

Lastly, define $\mathcal{X}_F = \{x_n |$ there exists feasible $\mathcal{M}_n\}$. It must be that $\Omega \subseteq \mathcal{X}_F$, and in general $\mathcal{X}_F$ will be larger. There is in fact a tradeoff between the values of N and the size of the set $\mathcal{X}_F$. Feasibility requires being able to steer the system into Ω at time N. The larger N is, the more initial conditions can be steered to Ω and so $\mathcal{X}_F$ will be larger. However, having a larger N means there are more variables and constraints in our optimization problem, and so we will require more computation for larger values of N.

6 Design Variables in Linear Tube MPC

Consider the linear tube MPC formulation:

$$
\begin{aligned}
V_n(x_n) = \min & \psi_n(\overline{x}_n, \ldots, \overline{x}_{n+N}, \check{u}_n, \ldots, \check{u}_{n+N-1}) \\
\text{s.t. } & \overline{x}_{n+1+k} = A\overline{x}_{n+k} + B\check{u}_{n+k}, \forall k = 0, \ldots, N-1 \\
& \overline{x}_n = x_n \\
& \overline{x}_{n+k} \in \mathcal{X} \ominus \mathcal{R}_k, \forall k = 1, \ldots, N-1 \\
& \check{u}_{n+k} = K\overline{x}_{n+k} + c_k, \forall k = 0, \ldots, N-1 \\
& \check{u}_{n+k} \in \mathcal{U} \ominus K\mathcal{R}_k, \forall k = 0, \ldots, N-1 \\
& \overline{x}_{n+N} \in \Omega \ominus \mathcal{R}_N
\end{aligned}
$$

where $\mathcal{X}, \mathcal{U}, \Omega$ are polytopes, $\mathcal{R}_0 = \{0\}$, $\mathcal{R}_k = \oplus_{j=0}^{k-1}(A + BK)^j\mathcal{W}$ is a disturbance tube, and $N > 0$ is the horizon. Though it looks complex, there are only a few design choices. In particular, we engineer the following components:

- $\overline{x}_{k+1} = A\overline{x}_k + B\check{u}_k$ – We must construct a system model from prior knowledge about the system and experimental data.
- $\mathcal{X}$ – Constraints on the states come from information about the system.
- $\mathcal{U}$ – Constraints on the inputs come from information about the system.
- $\mathcal{W}$ – Disturbance bounds come from information about the system.
- $Q > 0$, $R > 0$ – These are tuning parameters that manage the tradeoff between control effort (higher R penalizes strong control effort) and state deviation (higher Q more strongly penalizes state deviation from the equilibrium point).
- N – The horizon length manages the tradeoff between performance of the controller versus computational effort (higher N requires more computation but the resulting controller has better performance).

The other elements are algorithmically computed based on the design choices:

- K – This is computed by the solution to an infinite horizon LQR problem.

- $P > 0$ – This is computed by the solution to an infinite horizon LQR problem.
- Ω – This set is computed by using the K that solves the infinite horizon LQR problem, in conjunction with an existing algorithm.
- $\mathcal{R}_k$ – These sets are computed using standard polytope algorithms for linear transformations and Minkowski sums of polytopes.

It is worth mentioning that there are more exotic MPC schemes that introduce more design parameters, in exchange for better performance.