# IEOR 265 – Lecture 10
# Observability

## 1 Definitions

Consider a discrete time LTI system:

$$x_{n+1} = Ax_n + Bu_n, \quad x_0 = \xi.$$

Now suppose that we do not measure $x_n$. Instead, consider a model in which we measure $u_n$ and

$$y_n = Cx_n + Du_n,$$

where $C \in \mathbb{R}^{m\times p}$ and $D \in \mathbb{R}^{m\times q}$; this equation is often called a read-out equation. Note that we assume that $A \in \mathbb{R}^{p\times p}$, $B \in \mathbb{R}^{p\times q}$, $C \in \mathbb{R}^{m\times p}$, and $D \in \mathbb{R}^{m\times q}$ are known. We have two related definitions. The LTI system with pair $(C, A)$ is:

1. *observable* if and only if given values of $u_n$ and $y_n$ for $n = 0, \ldots, p-1$, we can uniquely determine $x_0$;
2. *detectable* if and only if given values of $u_n$ and $y_n$ for $n = 0, \ldots, p-1$, we can determine an estimate $\hat{x}_n$ such that $\|x_n - \hat{x}_n\| \to 0$.

These definitions are related because if an LTI system is observable, then it is also detectable. The converse is not true: There are detectable LTI systems that are not observable. Also not that we made these definitions only with respect to the pair $(C, A)$ and not $B$ or $D$. We can remove the effect of $D$ by considering a model $\check{y}_n = y_n - Du_n = Cx_n$. And because $x_n = A^n x_0 + \sum_{k=0}^{n-1} A^{n-k-1}Bu_k$, we can subtract out the $B$ by defining $\overline{y}_n = \check{y}_n - C\sum_{k=0}^{n-1} A^{n-k-1}Bu_k = CA^n x_0$. Lastly, note that these definitions do not say anything about boundedness of the states. We could in fact have that $\|x_n\| \to \infty$.

## 2 Conditions

There is a duality between controllability (stabilizability) and observability (detectability). A pair $(C, A)$ is observable if and only if the pair $(A', C')$ is controllable. Similarly, a pair $(C, A)$ is detectable if and only if the pair $(A', C')$ is stabilizable.

## 3 Linear Observer

The concepts of observability and detectability are important because of the following result: An LTI system $(C, A)$ is detectable if and only if there exists a constant matrix $L \in \mathbb{R}^{p\times m}$ such that

$A + LC$ is stable. To understand why this is relevant, suppose that we choose the following estimate

$$\begin{aligned} \hat{x}_{n+1} &= A\hat{x}_n + Bu_n + L(\hat{y}_n - y), \quad \hat{x}_0 = \phi \\ \hat{y}_n &= C\hat{x}_n + Du_n. \end{aligned}$$

Now if we define the estimation error as $e_n = \hat{x}_n - x_n$, then we have

$$\begin{aligned} e_{n+1} = \hat{x}_{n+1} - x_{n+1} &= A\hat{x}_n + Bu_n + L(C\hat{x}_n + Du_n - Cx_n - Du_n) - Ax_n - Bu_n \\ &= (A + LC)(\hat{x}_n - x_n) = (A + LC)e_n, \end{aligned}$$

meaning that $\|e_n\| = \|\hat{x}_n - x_n\| \to 0$ because $A + LC$ is stable.

The condition of observability is even more powerful. Let $\lambda_1, \lambda_2, \ldots, \lambda_p \in \mathbb{C}$ be fixed complex numbers. If $(C, A)$ is observable, then there exists an $L$ such that the eigenvalues of $A+LC$ are precisely the $\lambda_1, \lambda_2, \ldots, \lambda_p$ that were chosen.

# 4 Steady State Kalman Filter

Consider the following LTI system with noise:

$$\begin{aligned} x_{n+1} &= Ax_n + v_n \\ y_n &= Cx_n + w_n \end{aligned}$$

where $v_n \sim \mathcal{N}(0, Q)$ is process noise (or state noise) and $w_n \sim \mathcal{N}(0, R)$ is measurement noise. The initial condition to this system is $x_0 \sim \mathcal{N}(\mu, \Sigma_0)$. For simplicity, we will assume that $Q > 0$ and $R > 0$.

Based on this system, consider the following optimization problem

$$\begin{aligned} \lim_{n\to\infty} \min\ & \mathbb{E}\Big[(\hat{x}_{n+1} - x_{n+1})'(\hat{x}_{n+1} - x_{n+1})\Big] \\ \text{s.t. }\ & x_{k+1} = Ax_k + v_k \\ & y_k = Cx_k + w_k \\ & v_k \sim \mathcal{N}(0, Q) \\ & w_k \sim \mathcal{N}(0, R) \end{aligned}$$

Note that this minimum may not be finite unless we impose additional restrictions.

In particular, suppose that $(C, A)$ is detectible. Then the minimizer is given by $\hat{x}_{n+1} = A\hat{x}_n + L(\hat{y}_n - y)$ (i.e., linear observer with constant gain), where

$$L = -APC'(R + CPC')^{-1}$$

and $P > 0$ is the unique solution to the discrete time algebraic Riaccati equation (DARE)

$$P = Q + A(P - PC'(R + CPC')^{-1}CP)A'.$$

If $K$ is the feedback gain for the infinite horizon LQR problem with pair $(A', C')$, then we actually have that $K = L'$; in other words, there is a duality between the infinite horizon LQR problem and the steady-state Kalman filter gain.

# 5 Separation Principle

Suppose we have an LTI system in which $(A, B)$ is stabilizable and $(C, A)$ is detectable. And imagine that we do not have access to measurements of $x_n$, rather we only measure $u_n$ and $y_n$. An interesting question to what happens if we use an observer to produce estimates $\hat{x}_n$, and then uses these estimates with a linear feedback to control the system? Is the resulting closed-loop system stable? It turns out that the answer is yes, and the answer lets us separate the observer design from the controller design.

In particular, consider an output-feedback controller

$$\begin{aligned} \hat{x}_{n+1} &= A\hat{x}_n + Bu_n + L(C\hat{x}_n + Du_n - y_n) \\ u_n &= K\hat{x}_n, \end{aligned}$$

where $K, L$ are any matrices such that $(A + BK)$ and $(A + LC)$ are stable. Note that the closed-loop system is given by

$$\begin{bmatrix} x_{n+1} \\ \hat{x}_{n+1} \end{bmatrix} = \begin{bmatrix} A & BK \\ -LC & A + BK + LC \end{bmatrix} \begin{bmatrix} x_n \\ \hat{x}_n \end{bmatrix}.$$

Next consider a change of variables

$$\begin{bmatrix} x_n \\ e_n \end{bmatrix} = \begin{bmatrix} \mathbb{I} & 0 \\ -\mathbb{I} & \mathbb{I} \end{bmatrix} \begin{bmatrix} x_n \\ \hat{x}_n \end{bmatrix}.$$

Then the dynamics in this new coordinate system are given by

$$\begin{aligned} \begin{bmatrix} x_{n+1} \\ e_{n+1} \end{bmatrix} &= \begin{bmatrix} \mathbb{I} & 0 \\ -\mathbb{I} & \mathbb{I} \end{bmatrix} \begin{bmatrix} A & BK \\ -LC & A + BK + LC \end{bmatrix} \left( \begin{bmatrix} \mathbb{I} & 0 \\ -\mathbb{I} & \mathbb{I} \end{bmatrix} \right)^{-1} \begin{bmatrix} x_n \\ e_n \end{bmatrix} \\ &= \begin{bmatrix} A & BK \\ -LC - A & A + LC \end{bmatrix} \begin{bmatrix} \mathbb{I} & 0 \\ \mathbb{I} & \mathbb{I} \end{bmatrix} \begin{bmatrix} x_n \\ e_n \end{bmatrix} \\ &= \begin{bmatrix} A + BK & BK \\ 0 & A + LC \end{bmatrix} \begin{bmatrix} x_n \\ e_n \end{bmatrix} \end{aligned}$$

The eigenvalues of this block matrix are precisely the eigenvalues of $A + BK$ and $A + LC$, and so the closed-loop system as long as $A + BK$ and $A + LC$ are both individually stable.