# IEOR 165 – Lecture 11
# Semiparametric Models

## 1 Kernel Estimators

### 1.1 Convergence Rate

There is one point of caution to note regarding the use of kernel density estimation (and any other nonparametric density estimators like the histogram). Suppose we have data $x_i$ from a multivariate jointly Gaussian distribution $\mathcal{N}(\mu, \Sigma)$. The we can use the sample mean vector estimate

$$\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

and sample covariance matrix estimate

$$\hat{\Sigma} = \frac{1}{n}\sum_{i=1}^{n}(x_i - \hat{\mu})(x_i - \hat{\mu})'.$$

to estimate the distribution as $\mathcal{N}(\hat{\mu}, \hat{\Sigma})$. The estimation error of this approach is $O_p(d/n)$, and so the error increases linearly in dimension. In contrast, the estimation error of the kernel density estimate is $O_p(n^{-2/(d+4)})$. This means that the estimation error is exponentially worse in terms of dimension $d$, and this is an example of the *curse of dimensionality* in the statistics context. In fact, the estimation error of the kernel density estimate is typically $O_p(n^{-2/(d+4)})$ when applied to a general distribution.

### 1.2 Nadaraya-Watson Estimator

Consider the nonlinear model $y_i = g(x_i) + \epsilon_i$, where $g(\cdot)$ is an unknown nonlinear function. Suppose that given $x_0$, we would like to only estimate $g(x_0)$. One estimator that can be used is

$$\hat{g}(x_0) = \frac{\sum_{i=1}^{n} K(\|x_i - x_0\|/h) \cdot y_i}{\sum_{i=1}^{n} K(\|x_i - x_0\|/h)},$$

where $K(\cdot)$ is a kernel function. This estimator is known as the Nadaraya-Watson estimator, and it was one of the earlier techniques developed for nonparametric regression.

### 1.3 Example: Telephone Call Data

Suppose the lengths of calls at a call center are

$$x_i = \{0.66, 0.05, 0.27, 1.26, 1.51, 0.38, 1.79, 0.94, 0.48, 0.89\}.$$

And imagine that we conduct a survey after each call where we ask the customer to rate their satisfaction with the call. Suppose the corresponding satisfaction levels (1 = very dissatisfied, 2 = somewhat dissatisfied, 3 = neutral, 4 = somewhat satisfied, and 5 = very satisfied) are

$$y_i = \{3, 5, 4, 1, 1, 3, 2, 5, 4, 2\}.$$

Q: Suppose we choose $h = 0.5$, and that we use the Epanechnikov kernel. Estimate the satisfaction level for a telephone call of length 0.7 using the Nadaraya-Watson estimator.

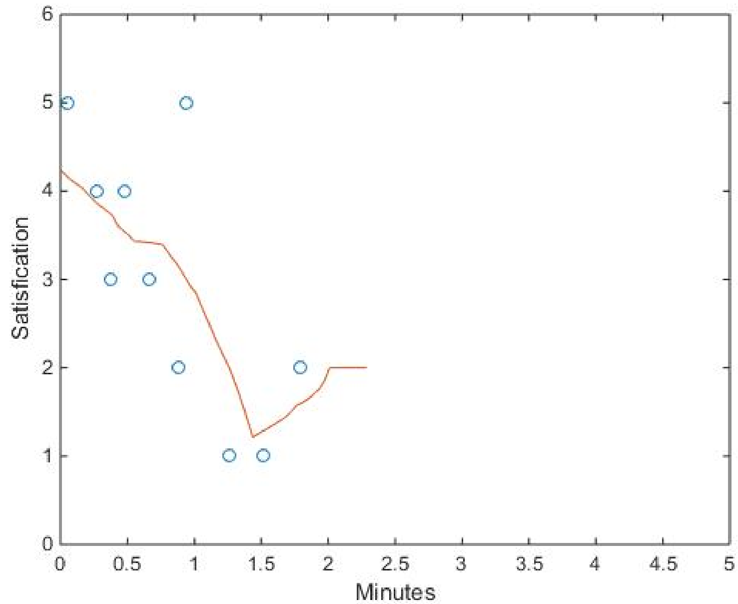
A: We first compute the quantities $K\left(\frac{u-x_i}{h}\right)$ for each data point. We have

$$\begin{aligned}
&K((0.7-0.66)/0.5) = K(0.08) = 3/4 \cdot (1-0.08^2) = 0.7452\\
&K((0.7-0.05)/0.5) = K(1.3) = 0\\
&K((0.7-0.27)/0.5) = K(0.86) = 3/4 \cdot (1-0.86^2) = 0.1953\\
&K((0.7-1.26)/0.5) = K(-1.12) = 0\\
&K((0.7-1.51)/0.5) = K(-1.62) = 0\\
&K((0.7-0.38)/0.5) = K(0.64) = 3/4 \cdot (1-0.64^2) = 0.4428\\
&K((0.7-1.79)/0.5) = K(-2.18) = 0\\
&K((0.7-0.94)/0.5) = K(-0.48) = 3/4 \cdot (1-0.48^2) = 0.5772\\
&K((0.7-0.48)/0.5) = K(0.44) = 3/4 \cdot (1-0.44^2) = 0.6048\\
&K((0.7-0.89)/0.5) = K(-0.38) = 3/4 \cdot (1-0.38^2) = 0.6417
\end{aligned}$$

Finally, we compute

$$\begin{aligned}
\hat{g}(0.7) = \frac{\sum_{i=1}^n K(\|x_i - x_0\|/h) \cdot y_i}{\sum_{i=1}^n K(\|x_i - x_0\|/h)} = \\
\frac{0.7452 \cdot 3 + 0.1953 \cdot 4 + 0.4428 \cdot 3 + 0.5772 \cdot 5 + 0.6048 \cdot 4 + 0.6417 \cdot 2}{0.7452 + 0.1953 + 0.4428 + 0.5772 + 0.6048 + 0.6417} = 3.41.
\end{aligned}$$

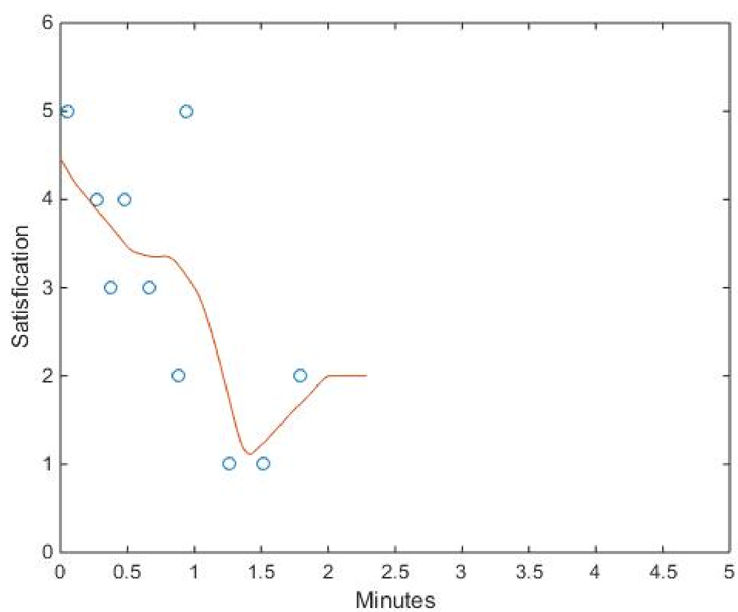
For reference, the scatter plot and the full curve $\hat{g}(x)$ estimated by Nadaraya-Watson are shown below. The full curve is generally calculated using a computer.




The Epanechnikov kernel is not differentiable, and so the estimated curve is not differentiable. An example of a kernel function that is differentiable is the biweight kernel, which is defined as

$$K(u) = \frac{15}{16} \cdot (1 - u^2)^2 \mathbf{1}(|u| \leq 1).$$

If we use the biweight kernel, then the full curve estimated by Nadaraya-Watson is




This curve is differentiable.

## 1.4 Small Denominators in Nadaraya-Watson

The denominator of the Nadaraya-Watson estimator is worth examining. Define

$$\hat{g}(x_0) = \frac{1}{nh^p} \sum_{i=1}^{n} K(\|x_i - x_0\|/h),$$

and note that $\hat{f}(x_0)$ is an estimate of the probability density function of $x_i$ at the point $x_0$. This is known as a kernel density estimate (KDE), and the intuition is that this is a smooth version of a histogram of the $x_i$.

The denominator of the Nadaraya-Watson estimator is a random variable, and technical problems occur when this denominator is small. This can be visualized graphically. The traditional approach to dealing with this is *trimming*, in which small denominators are eliminated. The trimmed version of the Nadaraya-Watson estimator is

$$\hat{g}(x_0) = \begin{cases} \frac{\sum_{i=1}^{n} K(\|x_i - x_0\|/h) \cdot y_i}{\sum_{i=1}^{n} K(\|x_i - x_0\|/h)}, & \text{if } \sum_{i=1}^{n} K(\|x_i - x_0\|/h) > \mu \\ 0, & \text{otherwise} \end{cases}.$$

One disadvantage of this approach is that if we think of $\hat{g}(x_0)$ as a function of $x_0$, then this function is not differentiable in $x_0$.

## 1.5 Example: Telephone Call Data

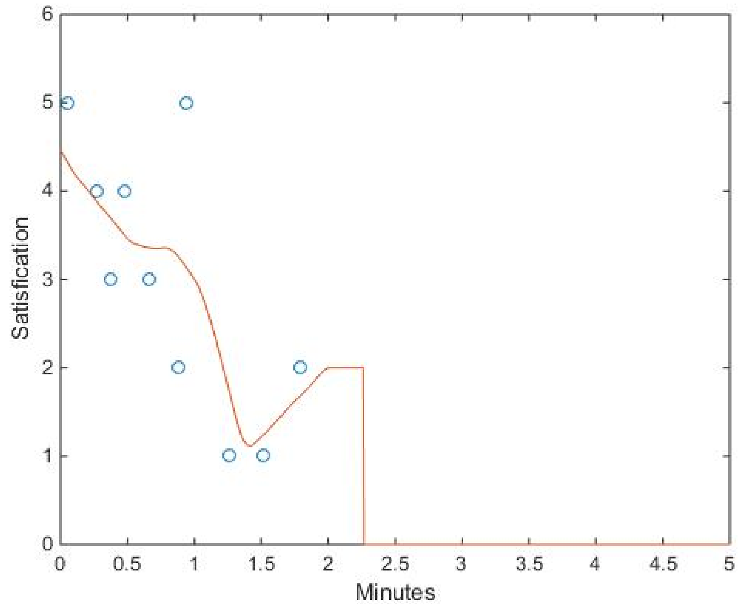
Suppose the lengths of calls at a call center are

$$x_i = \{0.66, 0.05, 0.27, 1.26, 1.51, 0.38, 1.79, 0.94, 0.48, 0.89\}.$$

And imagine that we conduct a survey after each call where we ask the customer to rate their satisfaction with the call. Suppose the corresponding satisfaction levels (1 = very dissatisfied, 2 = somewhat dissatisfied, 3 = neutral, 4 = somewhat satisfied, and 5 = very satisfied) are

$$y_i = \{3, 5, 4, 1, 1, 3, 2, 5, 4, 2\}.$$

Then the trimmed Nadraya-Watson estimator using the biweight kernel with bandwidth $h = 0.5$ and threshold $\mu = 0.01$ is:




## 1.6 $L2$-Regularized Nadaraya-Watson Estimator

A new approach is to define the $L2$-regularized Nadaraya-Watson estimator

$$\hat{g}(x_0) = \frac{\sum_{i=1}^{n} K(\|x_i - x_0\|/h) \cdot y_i}{\lambda + \sum_{i=1}^{n} K(\|x_i - x_0\|/h)},$$

where $\lambda > 0$. If the kernel function is differentiable, then the function $\hat{g}(x_0)$ is always differentiable in $x_0$. The reason for the name of this estimator is that we have

$$\hat{g}(x_0) = \arg\min_{\beta_0} \|W_h^{1/2}(Y - 1_n\beta_0)\|_2^2 + \lambda\|\beta_0\|_2^2 = \arg\min_{\beta_0} \sum_{i=1}^{n} K(\|x_i - x_0\|/h) \cdot (y_i - \beta_0)^2 + \lambda\beta_0^2.$$

Lastly, note that we can also interpret this estimator as the mean with weights

$$\{\lambda, K(\|x_1 - x_0\|/h), \ldots, K(\|x_n - x_0\|/h)\}$$

of points $\{0, y_1, \ldots, y_n\}$.

## 1.7 Example: Telephone Call Data

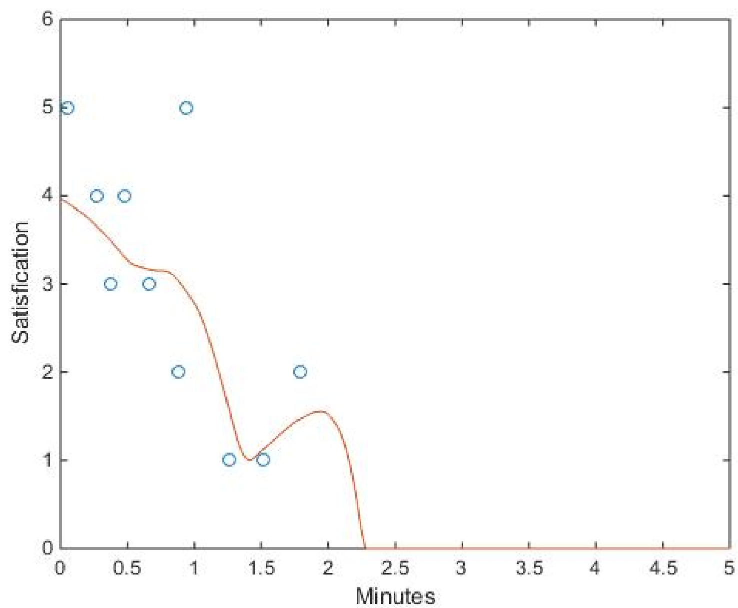
Suppose the lengths of calls at a call center are

$$x_i = \{0.66, 0.05, 0.27, 1.26, 1.51, 0.38, 1.79, 0.94, 0.48, 0.89\}.$$

And imagine that we conduct a survey after each call where we ask the customer to rate their satisfaction with the call. Suppose the corresponding satisfaction levels (1 = very dissatisfied, 2 = somewhat dissatisfied, 3 = neutral, 4 = somewhat satisfied, and 5 = very satisfied) are

$$y_i = \{3, 5, 4, 1, 1, 3, 2, 5, 4, 2\}.$$

Then the trimmed Nadraya-Watson estimator using the biweight kernel with bandwidth $h = 0.5$ and regularization $\lambda = 0.2$ is:




This curve is differentiable

# 2 Partially Linear Model

Consider the following model

$$y_i = x_i'\beta + g(z_i) + \epsilon_i,$$

where $y_i \in \mathbb{R}$, $x_i, \beta \in \mathbb{R}^p$, $z_i \in \mathbb{R}^q$, $g(\cdot)$ is an unknown nonlinear function, and $\epsilon_i$ are noise. The data $x_i, z_i$ are i.i.d., and the noise has conditionally zero mean $\mathbb{E}[\epsilon_i|x_i, z_i] = 0$ with unknown and bounded conditional variance $\mathbb{E}[\epsilon_i^2|x_i, z_i] = \sigma^2(x_i, z_i)$. This is known as a partially linear model because it consists of a (parametric) linear part $x_i'\beta$ and a nonparametric part $g(z_i)$. One can think of the $g(\cdot)$ as an infinite-dimensional nuisance parameter.

## 2.1 Semiparametric Approach

Ideally, our estimates of $\beta$ should converge at the parametric rate $O_p(1/\sqrt{n})$, but the $g(z_i)$ term causes difficulties in being able to achieve this. But if we could somehow subtract out this term, then we would be able to estimate $\beta$ at the parametric rate. This is the intuition behind the semiparametric approach. Observe that

$$\mathbb{E}[y_i|z_i] = \mathbb{E}[x_i'\beta + g(z_i) + \epsilon_i|z_i] = \mathbb{E}[x_i|z_i]'\beta + g(z_i),$$

and so

$$y_i - \mathbb{E}[y_i|z_i] = (x_i'\beta + g(z_i) + \epsilon_i) - \mathbb{E}[x_i|z_i]'\beta - g(z_i) = (x_i - \mathbb{E}[x_i|z_i])'\beta + \epsilon_i.$$

Now if we define

$$\hat{Y} = \begin{bmatrix} \mathbb{E}[y_1|z_1] \\ \vdots \\ \mathbb{E}[y_n|z_n] \end{bmatrix}$$

and

$$\hat{X} = \begin{bmatrix} \mathbb{E}[x_1|z_1]' \\ \vdots \\ \mathbb{E}[x_n|z_n]' \end{bmatrix}$$

then we can define an estimator

$$\hat{\beta} = \arg\min_{\beta} \|(Y - \hat{Y}) - (X - \hat{X})\beta\|_2^2 = ((X - \hat{X})'(X - \hat{X}))^{-1}((X - \hat{X})'(Y - \hat{Y})).$$

The only question is how can we compute $\mathbb{E}[x_i|z_i]$ and $\mathbb{E}[y_i|z_i]$? It turns out that if we compute those values with the trimmed version of the Nadaraya-Watson estimator, then the estimate $\hat{\beta}$ converges at the parametric rate under reasonable technical conditions. Intuitively, we would expect that we could alternatively use the $L2$-regularized Nadaraya-Watson estimator, but this has not yet been proven to be the case.

## 2.2 Example: Telephone Call Data

Suppose the lengths of calls at a call center are

$$x_i = \{0.66, 0.05, 0.27, 1.26, 1.51, 0.38, 1.79, 0.94, 0.48, 0.89\}.$$

And imagine that we conduct a survey after each call where we ask the customer to rate their satisfaction with the call. Suppose the corresponding satisfaction levels (1 = very dissatisfied, 2 = somewhat dissatisfied, 3 = neutral, 4 = somewhat satisfied, and 5 = very satisfied) are

$$y_i = \{3, 5, 4, 1, 1, 3, 2, 5, 4, 2\}.$$

Furthermore, suppose we also record the time of day for each call:

$$t_i = \{18, 19, 17, 13, 11, 19, 16, 12, 16, 10\}.$$

Now imagine that we believe that the model relating the satisfaction level to the length of call is

$$y = m \cdot x + g(t),$$

where $m$ is an unknown constant, and $g(\cdot)$ is an unknown function. Suppose we are interested in estimating $m$, which gives the sensitivity of satisfaction to the length of call. Then, one natural approach is to use semiparametric estimation.

Suppose we use the L2-regularized Nadaraya-Watson estimator with an Epanechnikov kernel, $h = 0.5$, and $\lambda = 0.2$. Then we get

$$\begin{aligned} \hat{x}_i &= \{0.5204, 0.1902, 0.2110, 0.9982, 1.1916, 0.1902, 1.0015, 0.7384, 1.0015, 0.7002\} \\ \hat{y}_i &= \{2.3684, 3.5294, 3.1579, 0.7895, 0.7895, 3.5294, 2.6471, 3.9474, 2.6471, 1.5789\}. \end{aligned}$$

Computing $\tilde{x}_i = x_i - \hat{x}_i$ and $\tilde{y}_i = y_i - \hat{y}_i$, we get

$$\begin{aligned} \tilde{x}_i &= \{0.1388, -0.1357, 0.0563, 0.2662, 0.3178, 0.1864, 0.7874, 0.1969, -0.5203, 0.1867\} \\ \tilde{y}_i &= \{0.6316, 1.4706, 0.8421, 0.2105, 0.2105, -0.5294, -0.6471, 1.0526, 1.3529, 0.4211\}. \end{aligned}$$

In this case, $\hat{m} = ((X - \hat{X})'(X - \hat{X}))^{-1}(X - \hat{X})'(Y - \hat{Y}) = \frac{\overline{\tilde{x}\tilde{y}}}{\overline{\tilde{x}^2}}$. Computing these quantities, we have:

$$\begin{aligned} \overline{\tilde{x}^2} &= 0.1212 \\ \overline{\tilde{x}\tilde{y}} &= -0.0968. \end{aligned}$$

Thus, we get

$$\hat{m} = \frac{\overline{\tilde{x}\tilde{y}}}{\overline{\tilde{x}^2}} = \frac{-0.0968}{0.1212} = -0.80.$$

For reference, if we had identified a model

$$y = m \cdot x + b,$$

then the estimate would have been $\hat{m} = -1.92$ and $\hat{b} = 4.6$. In this example, adjusting the model for the time of day makes a significant difference in our estimate.