IEOR 165 – Practice Questions for Midterm

1. Suppose we have measured pairs of price for a single can versus number of pop cans purchased. The data of price for a single can is $1.00, 1.20, 1.10, 1.15, 1.05, 0.95$ and the data for number of pop cans purchased is $30, 20, 30, 31, 36, 40$. Construct a linear model to predict the number of purchased pop cans as a function of the price for a single can. Let p be price for a single can of pop, and let d be the number of pop cans purchased.

   Answer: $d = -58.9 \cdot p + 94.4$

2. For the previous problem, let p be price for a single can and supose $\mathbb{E}[\epsilon^2|p] = p$. Use weighted least squares to construct a linear model to predict the number of purchased pop cans as a function of the price for a single can.

   Answer: $\cancel{d = -45.3 \cdot p + 80.8}$ $d = 21.6p + 8.51$

3. Suppose the amount of electricity through a scientific equipment is exponential in distance. That is, consider a functional form of $I = k \cdot \exp(x)$, where I is current, x is distance, and k is a parameter. If we measure distances $7.1, 0.3, 2.8, 0.5, 1.0$ with corresponding currents $2300, 3, 32, 5, 53$, then estimate the parameter k.

   Answer: $\hat{k} = 1.9$

4. Consider a random variable $Z_i = i \cdot \theta \cdot X_i$, where $X_i \sim \mathcal{N}(0, 1)$ are i.i.d. Given data Z_i for $i = 1, \ldots, n$, what is the ~~MAP~~ MLE estimate for θ^2?

   Answer: $\hat{\theta^2} = \frac{1}{n}\sum_{i=1}^{n}(Z_i^2/i^2)$

5. Suppose $X \sim \mathcal{N}(\mu, 1)$, and that $\mathbb{P}(\mu = +1) = 1/2$ and $\mathbb{P}(\mu = -1) = 1/2$. What is the MAP estimate for μ using a single data point X?

   Answer: $\hat{\mu} = +1$ if $X > 0$ and $\hat{\mu} = -1$ if $X < 0$. If $X = 0$, then the MAP is nonunique: Either $\hat{\mu} = \pm 1$ is correct.

1. $\bar{p} = \frac{1}{6}(1.00 + 1.20 + 1.10 + 1.15 + 1.05 + 0.95) = 1.0750$

$\bar{d} = \frac{1}{6}(30 + 20 + 30 + 31 + 36 + 40) = 31.1667$

$\overline{pd} = \frac{1}{6}(1.00 \cdot 30 + 1.20 \cdot 20 + 1.10 \cdot 30 + 1.15 \cdot 31 + 1.05 \cdot 36 + 0.95 \cdot 40) = 33.0750$

$\overline{p^2} = \frac{1}{6}(1.00^2 + 1.20^2 + 1.10^2 + 1.15^2 + 1.05^2 + 0.95^2) = 1.1629$

$\hat{m} = (\overline{pd} - \bar{p} \cdot \bar{d}) / (\overline{p^2} - (\bar{p})^2) = (33.0750 - 1.0750 \cdot 31.1667) / (1.1629 - 1.0750^2)$

$= -58.9$

$\hat{k} = \bar{d} - \hat{m}\bar{p} = 31.1667 + 58.9 \cdot 1.075$

$= 94.4$

$\boxed{d = -58.9p + 94.4}$

2. $w = \frac{1}{p}$, so w is $1.0000, 0.8333, 0.9091, 0.8696, 0.9524, 1.0526$

$\overline{wp} = \frac{1}{6}(1.00 \cdot 1.0000 + 1.20 \cdot 0.8333 + 1.10 \cdot 0.9091 + 1.15 \cdot 0.8696 + 1.05 \cdot 0.9524 + 0.95 \cdot 1.0526)$
$= 1.0000$

$\overline{wd} = 29.5478$

$\overline{wpd} = 31.1667$, $\overline{wp^2} = 1.0750$, $\overline{w} = 0.9362$

$\hat{m} = (\overline{wpd} - \overline{wp} \cdot \overline{wd}) / (\overline{wp^2} - (\overline{wp})^2) = 21.5853$

$\hat{k} = (\overline{wd} - \hat{m}\,\overline{wp}) / \overline{w} = 8.5051$

$\boxed{d = 21.6p + 8.51}$

3. $I = k\exp(x) \Rightarrow \mathbb{E}(I) = k\,\mathbb{E}(\exp(x))$

$\overline{I} = \frac{1}{5}(2300 + 3 + 32 + 5 + 53) = 478.6000$ and $\overline{\exp(x)} = \frac{1}{5}(\exp(7.1) + \exp(0.3) + \exp(2.8) + \exp(0.5) + \exp(1.0)) = 246.8257$

$\boxed{\hat{k} = \overline{I} / \overline{\exp(x)} = 1.9}$

4. Note that $z_i \sim N(0, i^2\theta^2)$. So $\hat{\theta} = \arg\max \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi i^2\theta^2}} \exp\left(\frac{-z_i^2}{2i^2\theta^2}\right)$

Taking the negative logarithm,

$$\hat{\theta} = \arg\min \sum_{i=1}^{n}\left(\frac{1}{2}\log(2\pi i^2\theta^2) + \frac{z_i^2}{2i^2\theta^2}\right)$$

Differentiating the objective with respect to θ^2

$$\sum_{i=1}^{n}\left(\frac{1}{2\theta^2} + \frac{-z_i^2}{2i^2(\theta^2)^2}\right) = 0 \Rightarrow n\theta^2 = \sum_{i=1}^{n}\frac{z_i^2}{i^2}$$

$$\Rightarrow \boxed{\hat{\theta}^2 = \frac{1}{n}\sum_{i=1}^{n}\frac{z_i^2}{i^2}}$$

5. The MAP is $\hat{\mu} = \arg\max \exp\left(\frac{-(x-1)^2}{2}\right) \cdot \mathbb{1}(\mu = +1) + \exp\left(-\frac{(x+1)^2}{2}\right) \cdot \mathbb{1}(\mu = -1)$

$$= \begin{cases} 1, & \text{if } \exp\left(-\frac{(x-1)^2}{2}\right) > \exp\left(-\frac{(x+1)^2}{2}\right) \\ -1, & \text{if } \exp\left(-\frac{(x+1)^2}{2}\right) < \exp\left(-\frac{(x+1)^2}{2}\right) \\ \pm 1, & \text{otherwise} \end{cases}$$

$$= \begin{cases} 1, & \text{if } x > 0 \\ -1, & \text{if } x < 0 \\ \pm 1, & \text{if } x = 0 \end{cases}$$