Department of Industrial Engineering & Operations Research

IEOR 165 (Spring 2017)

Homework 2

Due: Thursday, March 2

Question 1. Let $X_1, \ldots, X_n$ be iid from the pdf

$$f(x) = \theta^2 x e^{-\theta x}, \quad 0 < x,\ 0 < \theta < \infty$$

Find the MLE of θ.

Solution. The likelihood function is given as

$$\mathcal{L}(\theta) = (\prod_{i=1}^{n} x_i)\theta^{2n} \exp(-\theta \sum_{i=1}^{n} x_i) \tag{1}$$

So taking the log we get that the log likelihood is:

$$\ln \mathcal{L}(\theta) = 2n \ln \theta - \theta \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \ln x_i \tag{2}$$

Taking the derivative and setting to zero we get that $\hat{\theta}_{mle} = \frac{2n}{\sum_{i=1}^{n} x_i}$.

Question 2. Let $X_1, \ldots, X_n$ be iid with pmf

$$f_\theta(x) = \binom{N}{x} \theta^x (1-\theta)^{N-x}, \quad 0 \leq x \leq N, \quad 0 \leq \theta \leq 1$$

Where N is a known constant. Find the MLE of θ.

Solution. The likelihood function is:

$$\mathcal{L}(\theta) = (\prod_{i=1}^{n} \binom{N}{x_i})\theta^{\sum_{i=1}^{n} x_i}(1-\theta)^{Nn - \sum_{i=1}^{n} x_i} \tag{3}$$

Taking the log we get that the log likelihood is:

$$\ln \mathcal{L}(\theta) = \sum_{i=1}^{n} \ln \binom{N}{x_i} + (\ln \theta) \sum_{i=1}^{n} x_i + (\ln(1-\theta))(Nn - \sum_{i=1}^{n} x_i) \tag{4}$$

Taking the derivative and setting to zero we get that $\hat{\theta}_{mle} = \frac{\sum_{i=1}^{n} x_i}{Nn}$

Question 3. The chlorine residual (C) in a swimming pool at various times after being cleaned (T) is as given:

Time (hr)	Chlorine Residual (pt/million)
2	1.92
4	1.55
6	1.47
8	1.33
10	1.43
12	1.08

Assume the following relationship

$$C \approx a \exp(-bT)$$

What would you predict for the chlorine residual 15 hours after a cleaning?

Solution. Take log on both sides:

$$\log C \approx \log a - bT$$

So we derive a linear model (in approximation) with $y = \log C$ and $x = T$. Let $\beta_0 = \log a$ and $\beta_1 = -b$, we get the OLS estimates as

$$\hat{\beta}_0 \approx 0.69 \quad \hat{\beta}_1 \approx -0.046$$

So at $T = 15$,

$$\log(C(15)) = 0.69 - 0.046 \times 15 = 0.0$$

Thus $C(15) = e^{0.0} = 1.0$.

Question 4. Assume we have one observation X drawn from a Bernoulli distribution with unknown parameter p. p itself follows a beta distribution with shape parameters α and β. Show that the posterior distribution is beta and find its mean and variance ($\mathbb{E}[p|X]$ and $Var(p|X)$)
Hint 1: The pdf of a beat distribution with parameters α, β is:

$$f(y) = \frac{y^{\alpha-1}(1-y)^{\beta-1}}{\mathbf{B}(\alpha, \beta)} \tag{5}$$

Where $\mathbf{B}(a, b) = \frac{(a-1)!(b-1)!}{(a+b-1)!}$ is the beta function. The mean is $\frac{\alpha}{\alpha+\beta}$ and the variance is $\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$

Solution. By the Bayes' rule, we have

$$f(p|X) \propto f(X|p) \cdot f(p) = p^x(1-p)^{(1-x)} \cdot \frac{p^{\alpha-1}(1-p)^{\beta-1}}{\mathbf{B}(\alpha, \beta)} = \frac{p^{x+\alpha-1}(1-p)^{\beta-x}}{\mathbf{B}(\alpha, \beta)}$$

Normalizing this distribution we get that:

$$f(p|X) = \frac{p^{x+\alpha-1}(1-p)^{\beta-x}}{\mathbf{B}(x+\alpha, \beta-x+1)} \tag{6}$$

So using the formulas from the hint we see that $\mathbb{E}[p|X] = \frac{x+\alpha}{\alpha+\beta+1}$ and the variance is $\frac{(x+\alpha)(\beta-x+1)}{(\alpha+\beta+1)^2(\alpha+\beta+2)}$

Question 5. Find the Maximum *a posteriori* estimate (MAP) of p in Question 4.

Solution. Taking the log of the posterior we see that:

$$\ln f(p|X) = (x+\alpha-1)\ln p + (\beta - x)\ln(1-p) - \ln \mathbf{B}(x+\alpha, \beta - x + 1) \tag{7}$$

Taking the derivative and setting it equal to zero we see that $\hat{p}_{map} = \frac{1-(\alpha+x)}{1-(\beta+\alpha)}$.