IEOR 165 – Midterm
March 15, 2016

Name:	
Overall:	/50

Instructions:

1. Show your intermediate steps.
2. You are allowed a single 8.5x11 inch note sheet.
3. Calculators are allowed.

1	/10
2	/10
3	/10
4	/10
5	/10

1. It has been determined that the relation between stress (S) and the number of cycles to failure (N) for a particular type of alloy is given by

$$S = \frac{A}{N^m}$$

where A and m are unknown constants. An experiment is run yielding the following data.

Stress (thounsand psi)	N (million cycles to failure)
43.5	6.75
42.5	18.1
42.0	29.1
41.0	50.5
35.7	126
34.5	215
33.0	445
32.0	420

Use the above data to estimate A and m. (10 points)

Solution. Take the logarithm to both sides of the equation,

$$\log S = \log A - m \log N$$

Take $\log S$ as our response and $\log N$ as our predictor, then from OLS we have

$$\widehat{(\log A)} = 3.97 \qquad \widehat{-m} = -0.079$$

Hence the estimates of A and m are

$$\hat{A} = 52.98 \quad \hat{m} = 0.079$$

2. Suppose we make iid measurements of a random variable and get the following data set: $\{4, 5, 5, 6, 12, 14, 15, 15, 16, 17\}$. Use the kernel density estimator to estimate the pdf at $u = 15$, using the triangular kernel and a bandwidth of $h = 4$. (10 points)

   Hint: Recall that the triangluar kernel is

$$K(u) = \begin{cases} 1 - |u|, & \text{if } |u| \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

   Solution. First, we compute the kernel functions:

$$\begin{aligned} K((4-15)/4) &= 0 \\ K((5-15)/4) &= 0 \\ K((5-15)/4) &= 0 \\ K((6-15)/4) &= 0 \\ K((12-15)/4) &= 0.25 \\ K((14-15)/4) &= 0.75 \\ K((15-15)/4) &= 1 \\ K((15-15)/4) &= 1 \\ K((16-15)/4) &= 0.75 \\ K((17-15)/4) &= 0.5 \end{aligned}$$

   Then the kernel estimate is

$$\hat{f}^k(u = 15) = \frac{1}{nh} \sum_{i=1}^{n} K(\frac{x_i - u}{h}) = \frac{1}{10 \cdot 4}(0.25 + 0.75 + 1 + 1 + 0.75 + 0.5) = 0.10625$$

3. Suppose $x_1, \ldots, x_n$ are iid from a Bernuolli distribution, which means $\mathbb{P}(x_i = 1) = p$ and $\mathbb{P}(x_i = 0) = 1 - p$. Compute the MLE of p. (10 points)

Solution. First, the likelihood function is given as

$$L(p) = \prod_{i=1}^{n} \{\mathbf{1}(x_i = 1)p + \mathbf{1}(x_i = 0)(1-p)\} = p^m (1-p)^{n-m}$$

where m is the number of "1"s in the data. Then the log likelihood function is

$$l(p) = \log L(p) = m \log p + (n-m) \log(1-p)$$

Take the first derivative and set it equal to zero:

$$\frac{\partial l(p)}{\partial p} = \frac{m}{p} - \frac{n-m}{1-p} = 0 \Rightarrow \quad \hat{p} = \frac{m}{n}$$

4. Suppose our model is $y_i = \min\{\frac{x_i+\theta}{2}, 10\} + \epsilon_i$, where (i) ϵ_i has a Gaussian distribution with mean 0 and finite variance σ^2, and (ii) ϵ_i is independent of x_i, y_i. Using the iid data $(x_i, y_i) = \{(0,4),(0,6),(20,9),(20,11)\}$, compute the MLE estimate for θ. (10 points)

Solution. The likelihood function is

$$\begin{aligned} L(\theta) &= (\tfrac{1}{\sqrt{2\pi}\sigma})^4 \prod_{i=1}^4 \exp\big(-\tfrac{(y_i-\min\{\frac{x_i+\theta}{2},10\})^2}{2\sigma^2}\big) \\ &= (\tfrac{1}{\sqrt{2\pi}\sigma})^4 \exp\Big(-\tfrac{1}{2\sigma^2}\sum_{i=1}^4 (y_i - \min\{\tfrac{x_i+\theta}{2}, 10\})^2\Big) \end{aligned}$$

Then the log likelihood function is

$$l(\theta) = 4\log(\tfrac{1}{\sqrt{2\pi}\sigma}) - \tfrac{1}{2\sigma^2}\sum_{i=1}^4 (y_i - \min\{\tfrac{x_i+\theta}{2}, 10\})^2$$

As the first term in $l(\theta)$ is a constant, maximizing $l(\theta)$ equals to maximizing the second term, which is

$$\max_\theta \; -\sum_{i=1}^4 (y_i - \min\{\tfrac{x_i+\theta}{2}, 10\})^2$$

Now consider three cases:

- $\theta \geq 20$: the objective function is

  $$-[(4-10)^2 + (6-10)^2 + (9-10)^2 + (11-10)^2] = -54$$

  So the objective is -54.
- $0 \leq \theta < 20$: the objective function is

  $$-[(4-\tfrac{\theta}{2})^2 + (6-\tfrac{\theta}{2})^2 + (9-10)^2 + (11-10)^2]$$

  Setting the first derivative to zero: $-2(4-\frac{\theta}{2}) - 2(6-\frac{\theta}{2}) = 0$, implies $\theta = 10$. The objective function is $-(4-5)^2 + (6-5)^2 - (9-10)^2 - (11-10)^2 = -2$.
- $\theta < 0$: the objective function is

  $$-[(4-\tfrac{\theta}{2})^2 + (6-\tfrac{\theta}{2})^2 + (9-5-\tfrac{\theta}{2})^2 + (11-5-\tfrac{\theta}{2})^2]$$

  Its first derivative is: $-2(4-\frac{\theta}{2}) - 2(6-\frac{\theta}{2}) - 2(4-\frac{\theta}{2}) - 2(6-\frac{\theta}{2}) < 0$ for $\theta < 0$. This implies the objective is strictly decreasing as $\theta \to -\infty$ for $\theta < 0$. Thus, its maximum on this range is when $\theta = 0$, and the corresponding objective function value is $-(4-0)^2 + (6-0)^2 - (9-0)^2 - (11-0)^2 = -182$.

Comparing these values and we can conclude that the MLE estimate of θ is 10.

5. Suppose we have a prior distribution on θ (that only takes values of 0 or 1) of $\mathbb{P}(\theta = 0) = 1 - p$ and $\mathbb{P}(\theta = 1) = p$. Assume the conditional distribution of x given θ is a Gaussian distribution with mean $(-1)^\theta$ and variance of 1. Finally, suppose we only make a single measurement x_1. Compute the MAP estimate for θ as a function of x_1 and p. (10 points)

   Solution. First, we compute the posterior (drop the constant denominator)

$$
\begin{aligned}
p(x_1|\theta)g(\theta) &= \frac{1}{\sqrt{2\pi}} \exp(-\frac{(x_1 - (-1)^\theta)^2}{2})\{\mathbf{1}(\theta = 1)p + \mathbf{1}(\theta = 0)(1 - p)\} \\
&= \begin{cases} \frac{p}{\sqrt{2\pi}} \exp(-\frac{(x_1+1)^2}{2}) & \theta = 1 \\ \frac{1-p}{\sqrt{2\pi}} \exp(-\frac{(x_1-1)^2}{2}) & \theta = 0 \end{cases}
\end{aligned}
$$

   So the MAP estimate for θ is

$$
\hat{\theta} = \mathbf{1}\{p \cdot \exp(-\frac{(x_1 + 1)^2}{2})) \geq (1 - p) \exp(-\frac{(x_1 - 1)^2}{2})\}
$$