IEOR 265 – Lecture 18
Epi-Convergence

1 Sequence of Optimization Problems

Suppose we have an optimization problem

$$\min f(x),$$

where $f : \mathbb{R}^p \to \overline{\mathbb{R}}$ is lsc, level-bounded, and proper. In many applications, we may not be interested in directly solving this optimization problem. Instead, we will have a sequence of optimization problems that are meant to approximate this problem, and we instead solve these approximating optimization problems. This setting is not as abstract as it may seem at first glance: For example, this is implicitly what is done when using log-barrier functions, which are common in interior-point methods for numerically solving constrained optimization problems by converting a constrained problem into a sequence of approximating unconstrained problems. We will refer to this sequence of optimization problems as

$$\min f_n(x).$$

The central question with such sequences of approximating optimization problems is whether they provide a good approximation to the solution of the original optimization problem. That is, ideally we would like for $V_n \to V$, where

$$\begin{aligned} V_n &= \min f_n(x) \\ V &= \min f(x), \end{aligned}$$

and we would also like for $\limsup_n \mathcal{X}_n^* \subseteq \mathcal{X}^*$ where

$$\begin{aligned} \mathcal{X}_n^* &= \arg\min f_n(x) \\ \mathcal{X}^* &= \arg\min f(x). \end{aligned}$$

This last statement means that if x is any point such that there exists a subsequence $x_\nu^* \in \mathcal{X}_\nu^*$ such that $x_\nu^* \to x$, then $x \in \mathcal{X}^*$. The reason for this condition is that it may be the case that the limiting optimization problem has more minimizers than each individual optimization in the sequence. Note that when there is only a single minimizer, we must have that $x_n^* \to x^*$.

At its surface, this might seem like an easy exercise; however, certain pathologies can occur, as evidenced by the following example: Consider a sequence of optimization problems

$$\begin{aligned} \min\ & \min\{1-x, 1, 2n|x+1/n|-1\} \\ \text{s.t. } & x \in [-1, 1]. \end{aligned}$$

The objectives $f_n(x) = \min\{1-x, 1, 2n|x+1/n|-1\}$ converge pointwise to $f(x) = \min\{1-x, 1\}$ (recall that the definition of pointwise convergence is that $\lim_{n\to\infty} f_n(\overline{x}) = f(\overline{x})$ for all $\overline{x} \in \text{dom} f$). Next we turn our attention to the minimizers of the sequence, which are $x_n^* = -1/n$ and converge to zero. However, the minimizer of the optimization problem that we are approximating by the sequence is $x^* = 1$. The minimizers of the sequence do not converge to the minimizer of the limiting optimization problem.

Convergence of local minimizers to the local minimizers of a limiting optimization problem is more difficult because additional pathologies can occur. For instance, it is possible for local minimizers of a sequence of approximating problems to converge to the global maximizer of the limiting problem. Proving such pathologies does not occur requires the use of more sophisticated arguments, such as those using *optimality functions*.

2 Epi-convergence

It turns out that the right notion of convergence is *epi-convergence*, and the idea is that the epigraphs of the f_n converge to the epigraph of f (where convergence is defined in a particular way). One simpler characterization is that we say the f_n epi-converge to f (or $f_n \xrightarrow{e} f$) if and only if at each point x one has

$$\begin{cases} \liminf_n f_n(x_n) \geq f(x), \text{for every sequence } x_n \to x, \\ \limsup_n f_n(x_n) \leq f(x), \text{for some sequence } x_n \to x. \end{cases}$$

The reason this definition is useful is that there is a theorem which applies to the situation when the domain of f_n are eventually bounded, $f_n \xrightarrow{e} f$, f_n, f are lsc and proper. Under these conditions, we have that:

- $V_n \to V$ where V is finite;
- the sets $\arg\min f_n$ are nonempty and form a bounded sequence with

$$\limsup_n \mathcal{X}_n^* \subseteq \mathcal{X}^*.$$

3 Epi-convergence of Oracle in LBMPC

The reason these concepts are relevant to LBMPC is that the oracle $\mathcal{O}_n$ is changing. Consequently, we are not solving a fixed optimization; rather, we are solving a sequence of optimization problems. And it is natural to ask the question that if the oracle converges to the true model, then does the solution given by LBMPC converge to a solution that would be given if the optimization problem *a priori* knew the true model. This situation is more complicated than above, because we can have randomness in how we construct the oracle. As a result, the formal results require stochastic generalizations of the epi-convergence concepts discussed above.

In particular, suppose that there is a true model

$$x_{n+1} = Ax_n + Bu_n + g(x_n, u_n),$$

where $g(x_n, u_n) \in \mathcal{W}$. There are results for two important cases, corresponding to two classes of oracles.

- Parametric Oracle: If there exists λ_0 such that $g(x, u) = \chi(x, u, \lambda_0)$, then under some technical conditions we have that the control law of LBMPC with the parametric oracle $\mathcal{O}_n = \chi(x, u, \hat{\lambda})$ converges in probability to the control law of MPC that knows the true model.
- Nonparametric Oracle: If the nonparametric oracle converges uniformly in probability to the true modeling error

  $$\sup_{\mathcal{X}\times\mathcal{U}} \|\mathcal{O}_n(x, u) - g(x, u)\| = O_p(r_n),$$

  with $r_n \to 0$, then under some technical conditions the control law of LBMPC converges in probability to the control law of MPC that knows the true model.

4 Further Details

More details about the theory of epi-convergence can be found in the book *Variational Analysis* by Rockafellar and Wets, from which the material in the first two sections is found. Another useful reference is the book *Optimization: Algorithms and Consistent Approximations* by Polak.