IEOR 290A – Lecture 37
Estimating Multiple Utilities

1 Multiple Strategic Agents

Recall the following abstract model: Suppose that we have $p \geq 2$ agents, and the k-th agent is making decisions $x^{*,k} \in \mathcal{X}^k(u_i)$ to maximize their utility function while also taking into account the strategic behavior of each other agent. In this model, we need to also specify our notion of strategic behavior. Here, we will restrict out attention to the case of a Nash equilibrium, in which

$$x_i^{*,k} \in \arg\max\{J^k(x_i^{*,1}, \ldots, x_i^{*,k-1}, x^k, x_i^{*,k+1}, \ldots, x_i^{*,p}, u_i) \mid x^k \in \mathcal{X}^k(u_i)\}.$$

We observe $(u_i, x_i^{*,1}, \ldots, x_i^{*,p})$ for $i = 1, \ldots_n$ and would like to infer the utility function of each agent J^k.

To make this model more concrete, we will specify a specific instantiation of this problem. In particular, suppose that

- The constraint set $\mathcal{X}^k(u_i)$ is described by linear equality and inequality constraints:

  $$\mathcal{X}^k(u_i) = \{x : x \geq 0, F^k x + G^k u_i = h^k\},$$

  where (A^k, b^k) and (F^k, h^k) are suitably defined matrices and vectors.

- Assume that we have a parametrization of the utility function, that is we have

  $$\phi(x^1, \ldots, x^p, u; \beta),$$

  and a compact set Γ such that there exists $\beta_k \in \Gamma$ with

  $$J^k(x^1, \ldots, x^p, u) = \phi(x^1, \ldots, x^p, u; \beta_k).$$

Though these two conditions make the problem more specific, we will still impose additional conditions on the model formulation to make the problem computationally tractable.

2 Key Technical Difficulty

Consider the following feasibility problem formulation of the inverse decision-making problem for this multiple strategic agents model:

$$
\begin{aligned}
\hat{\beta} = \arg\min_{\beta}\ & 0 \\
\text{s.t. } & x_i^{*,k} \in \arg\max\{\phi(x_i^{*,1}, \ldots, x_i^{*,k-1}, x^k, x_i^{*,k+1}, \ldots, x_i^{*,p}, u_i; \beta) \mid \\
& \qquad\qquad\qquad\qquad\qquad\qquad x \geq 0, F^k x + G^k u_i = h^k\} \\
& \beta \in \Gamma.
\end{aligned}
$$

This feasibility problem is difficult to solve because it has an atypical constraint: The constraint that x_i^* be the equilibrium point of a game-theoretic model cannot be directly handled by nonlinear programming techniques. There are two reasons that this constraint presents challenges:

1. Depending on the value of β there may be one or multiple equilibria to the game.

2. The constraints have a complex form because it corresponds to equilibrium points to a game, and so we cannot in general hope for continuity of the equilibrium with respect to β, much less differentiablity.

3 Variational Inequalities

Similar to how KKT conditions represent optimality conditions for a nonlinear programming problem, there is an alternative representation for the Nash equilibrium of a game. Let

$$X = \begin{bmatrix} x^1 \\ \vdots \\ x^p \end{bmatrix}$$

and

$$\mathcal{X}(u) = \mathcal{X}^1(u) \times \cdots \times \mathcal{X}^p.$$

Assume that the utilities for each agent are differentiable, and define

$$D(X, u; \beta) = \begin{bmatrix} -\nabla_{x^1} \phi(X, u; \beta_1) \\ \vdots \\ -\nabla_{x^p} \phi(X, u; \beta_p) \end{bmatrix}.$$

Then the Nash equilibrium is also characterized by the following variational inequality:

$$D(X_i^*, u_i; \beta)'(X^* - X) \geq 0, \forall X \in \mathcal{X}(u_i).$$

Unfortunately, these are semi-infinite constraints because they must hold for all points $X \in \mathcal{X}(u_i)$. Because our constraints $\mathcal{X}^k(u_i)$ are such that they are a subset of $x \geq 0$, standard techniques from semi-infinite optimizaiton and from robust optimization can be used to show that satisfaction of the variational inequality is equivalent to the existence of y_i^k such that

$$
\begin{aligned}
F^{k\prime} y_i^k &\leq -\nabla_{x^k} \phi(X_i^*, u_i; \beta_k), \forall k = 1, \ldots, p \\
D(X_i^*, u_i; \beta)' X_i^* - \sum_{k=1}^{p} (h^k - G^k u_i)' y_i^k &\leq 0
\end{aligned}
$$

This characterization is better because it is only finite-dimensional (as opposed to semi-infinite).

4 Tractable Formulation

To make the problem more tractable, we will impose additional conditions on our model to ensure that the feasibility problem is convex. In particular, recall that the feasibility formulation is now given by

$$
\begin{aligned}
\hat{\beta} = \arg\min_{\beta} \; & 0 \\
\text{s.t. } & F^{k\prime} y_i^k \leq -\nabla_{x^k} \phi(X_i^*, u_i; \beta_k), \forall k = 1, \ldots, p \\
& D(X_i^*, u_i; \beta)' X_i^* - \sum_{k=1}^{p} (h^k - G^k u_i)' y_i^k \leq 0 \\
& \beta \in \Gamma.
\end{aligned}
$$

To ensure that these constraints are convex, we require that $\nabla_{x^k} \phi(X_i^*, u_i; \beta_k)$ is affine in β_k, for all k.

5 Further Details

More details about these concepts can be found in the paper *Data-Driven Estimation in Equilibrium Using Inverse Optimization* by Bertsimas, Gupta, and Paschalidis, from which the above material is found.