IEOR 290A – Lecture 22
(Robust) Linear Tube MPC

1 Optimization Formulation

Recall the LTI system with disturbance:

$$x_{n+1} = Ax_n + Bu_n + d_n,$$

where $d_n \in \mathcal{W}$. The issue with using a constant state-feedback controller $u_n = Kx_n$ is that the set Ω for which this controller is admissible and ensures constraint satisfaction subject to all possible sequences of disturbances (i.e., $d_0, d_1, \ldots$) can be quite small. In this sense, using this controller is conservative, and the relevant question is whether we can design a nonlinear controller $u_n = \ell(x_n)$ for the LTI system with disturbance such that constraints on states $x_n \in \mathcal{X}$ and inputs $u_n \in \mathcal{U}$ are satisfied. One idea is to use an MPC framework with robustness explicitly included.

(Robust) linear tube MPC is one approach for this. The idea underlying this approach is to subtract out the effect of the disturbance from our state and input constraints. One formulation is

$$\begin{aligned} V_n(x_n) = \min & \psi_n(\overline{x}_n, \ldots, \overline{x}_{n+N}, \check{u}_n, \ldots, \check{u}_{n+N-1}) \\ \text{s.t. } & \overline{x}_{n+1+k} = A\overline{x}_{n+k} + B\check{u}_{n+k}, \forall k = 0, \ldots, N-1 \\ & \overline{x}_n = x_n \\ & \overline{x}_{n+k} \in \mathcal{X} \ominus \mathcal{R}_k, \forall k = 1, \ldots, N-1 \\ & \check{u}_{n+k} = K\overline{x}_{n+k} + c_k, \forall k = 0, \ldots, N-1 \\ & \check{u}_{n+k} \in \mathcal{U} \ominus K\mathcal{R}_k, \forall k = 0, \ldots, N-1 \\ & \overline{x}_{n+N} \in \Omega \ominus \mathcal{R}_N \end{aligned}$$

where $\mathcal{X}, \mathcal{U}, \Omega$ are polytopes, $\mathcal{R}_0 = \{0\}$, $\mathcal{R}_k = \oplus_{j=0}^{k-1}(A + BK)^j\mathcal{W}$ is a disturbance tube, and $N > 0$ is the horizon. Note that we do not constrain the initial condition x_n, and this reduces the conservativeness of the MPC formulation. We will refer to the set Ω as a terminal set. The interpretation of the function ψ_n is a cost function on the states and inputs of the system.

2 Recursive Properties

Suppose that (A, B) is stabilizable and K is a matrix such that $(A + BK)$ is stable. For this given system and feedback controller, suppose we have a maximal output admissible disturbance invariant set Ω (meaning that this set has constraint satisfaction $\Omega \subseteq \{x : x \in \mathcal{X}, Kx \in \mathcal{U}\}$ and disturbance invariance $(A + BK)\Omega \oplus \mathcal{W} \subseteq \Omega$).

Next, note that conceptually our decision variables are c_k since the $\overline{x}_k, \check{u}_k$ are then uniquely determined by the initial condition and equality constraints. As a result, we will talk about solutions only in terms of the variables c_k. In particular, if $\mathcal{M}_n = \{c_n, \dots, c_{n+N-1}\}$ is feasible for the optimization defining linear MPC with an initial condition x_n, then the system that applies the control value $u_n = Kx_n + c_n[\mathcal{M}_n]$ results in:

- Recursive Feasibility: there exists feasible $\mathcal{M}_{n+1}$ for x_{n+1};
- Recursive Constraint Satisfaction: $x_{n+1} \in \mathcal{X}$.

We will give a sketch of the proof for these two results.

- Choose $\mathcal{M}_{n+1} = \{c_1[\mathcal{M}_n], \dots, c_{N-1}[\mathcal{M}_n], 0\}$. Some algebra gives that $\overline{x}_{n+1+k}[\mathcal{M}_{n+1}] = \overline{x}_{n+1+k}[\mathcal{M}_n] + (A+BK)^k d_n \in \overline{x}_{n+1+k}[\mathcal{M}_n] \oplus (A+BK)^k \mathcal{W}$. But because $\mathcal{M}_n$ is feasible, it must be that $\overline{x}_{n+1+k}[\mathcal{M}_n] \in \mathcal{X} \ominus \mathcal{R}_{k+1}$. Combining this gives

  $$\begin{aligned}\overline{x}_{n+1+k}[\mathcal{M}_{n+1}] \in \overline{x}_{n+1+k}[\mathcal{M}_n] &\oplus (A + BK)^k \mathcal{W} \\ &\subseteq \mathcal{X} \ominus (\mathcal{R}_k \oplus (A + BK)^k \mathcal{W}) \oplus (A + BK)^k \mathcal{W} \\ &\subseteq \mathcal{X} \ominus \mathcal{R}_k.\end{aligned}$$

  This shows feasibility for the states; a similar argument can be applied to the inputs.

  The final state needs to be considered separately. The argument above gives

  $$\overline{x}_{n+1+N-1}[\mathcal{M}_{n+1}] \in \Omega \ominus \mathcal{R}_{N-1}.$$

  Observe that the constraint satisfaction property of Ω leads to

  $$\check{u}_{n+1+N-1}[\mathcal{M}_{n+1}] = K\overline{x}_{n+1+N-1}[\mathcal{M}_{n+1}] \subseteq K\Omega \ominus K\mathcal{R}_{N-1} \subseteq \mathcal{U} \ominus K\mathcal{R}_{N-1}.$$

  Lastly, note that the disturbance invariance property of Ω gives

  $$\begin{aligned}\overline{x}_{n+1+N}[\mathcal{M}_{n+1}] \in &(A + BK)\Omega \ominus (A + BK)\mathcal{R}_{N-1} \\ &\subseteq (\Omega \ominus \mathcal{W}) \ominus (A + BK)\mathcal{R}_{N-1} = \Omega \ominus \mathcal{R}_N.\end{aligned}$$

- Since $x_{n+1} = \overline{x}_{n+1}[\mathcal{M}_n] + d_n \in (\mathcal{X} \ominus \mathcal{W}) \oplus \mathcal{W} \subseteq \mathcal{X}$, the result follows.

There is an immediate corollary of this theorem: If there exists feasible $\mathcal{M}_0$ for x_0, then the system is (a) Lyapunov stable, (b) satisfies all state and input constraints for all time, (c) feasible for all time.

Lastly, define $\mathcal{X}_F = \{x_n |$ there exists feasible $\mathcal{M}_n\}$. It must be that $\Omega \subseteq \mathcal{X}_F$, and in general $\mathcal{X}_F$ will be larger. There is in fact a tradeoff between the values of N and the size of the set $\mathcal{X}_F$. Feasibility requires being able to steer the system into Ω at time N. The larger N is, the more initial conditions can be steered to Ω and so $\mathcal{X}_F$ will be larger. However, having a larger N means there are more variables and constraints in our optimization problem, and so we will require more computation for larger values of N.