IEOR 290A – Lecture 15
Controllability

1 Definitions

Consider a discrete time LTI system:

$$x_{n+1} = Ax_n + Bu_n, \quad x_0 = \xi.$$

We have two related definitions. The LTI system defined by the pair (A, B) is:

1. *controllable* if and only if given any time $m \geq p + 1$ and any coordinate ϕ there exists a sequence of inputs $u_0, u_1, \dots, u_{m-1}$ such that $x_m = \phi$;
2. *stabilizable* if and only if there exists a sequence of inputs $u_0, u_1, \dots$ such that $\|x_n\| \to 0$.

These definitions are related because if an LTI system is controllable, then it is also stabilizable. The converse is not true: There are stabilizable LTI systems that are not controllable.

2 Conditions

We will describe several conceptual approaches to checking for controllability or stabilizability for (A, B).

1. Define the *controllability matrix*

   $$\mathcal{C} = \begin{bmatrix} B & AB & A^2B & \dots & A^{p-1}B \end{bmatrix}$$

   The pair (A, B) is controllable if and only if $\text{rank}(\mathcal{C}) = p$.

2. The *Popov–Belevitch–Hautus* (PBH) test is that (A, B) is controllable if and only if

   $$\text{rank}\left(\begin{bmatrix} s\mathbb{I} - A & B \end{bmatrix}\right) = p, \forall s \in \mathbb{C}.$$

   Furthermore, (A, B) is stabilizable if and only if

   $$\text{rank}\left(\begin{bmatrix} s\mathbb{I} - A & B \end{bmatrix}\right) = p, \forall s \in \mathbb{C} : |s| \geq 1.$$

3. Consider matrices A such that that $|(\sigma(A))_i| < 1$. The pair (A, B) is controllable if and only if the unique solution W to
$$AWA' - W = -BB'$$
is positive definite (i.e., $W > 0$). Observe that this is an LMI and can be solved using convex optimization approaches. Note that this W (if it exists) is equal to
$$W = \sum_{k=0}^{\infty} A^k BB'(A')^k,$$
which is known as the reachability Gramian.

4. The pair (A, B) is stabilizable if and only if there is a positive definite $P > 0$ solution to
$$APA' - P < BB'.$$
Note that this is an LMI and can be solved using convex optimization approaches.

3 Linear Feedback

The concepts of controllability and stabilizability are important because of the following result: An LTI system (A, B) is stabilizable if and only if there exists a constant matrix $K \in \mathbb{R}^{p\times q}$ such that choosing state-feedback input $u = Kx$ leads to a stable system
$$x_{n+1} = Ax_n + Bu_n = Ax_n + BKx_n = (A + BK)x_n,$$
meaning that the eigenvalues of $A + BK$ lie within the complex unit disc.

The condition of controllability is even more powerful. Let $\lambda_1, \lambda_2, \ldots, \lambda_p \in \mathbb{C}$ be fixed complex numbers. If (A, B) is controllable, then there exists a K such that the eigenvalues of $A + BK$ are precisely the $\lambda_1, \lambda_2, \ldots, \lambda_p$ that were chosen.

4 Finite Horizon Linear Quadratic Regulator (LQR)

Consider the following optimization problem
$$\min \left\{ \sum_{n=0}^{N} x_n' Q x_n + u_n' R u_n : x_{n+1} = Ax_n + Bu_n \right\},$$
where $Q > 0$ and $R > 0$ are positive definite matrices. The minimizer is given by $u_n = K_n x_n$, where
$$K_n = -(R + B'P_nB)^{-1}B'P_nA$$
and P_n is defined recursively by $P_N = Q$ and
$$P_{n-1} = Q + A'(P_n - P_nB(R + B'P_nB)^{-1}B'P_n)A.$$
The value function of this optimization is $V(x_0) = x_0' P_0 x_0$.

5 Infinite Horizon Linear Quadratic Regulator (LQR)

Consider the following optimization problem

$$\min \left\{ \sum_{n=0}^{\infty} x_n' Q x_n + u_n' R u_n : x_{n+1} = A x_n + B u_n \right\},$$

where $Q > 0$ and $R > 0$ are positive definite matrices. Note that this minimum may not be finite unless we impose additional restrictions.

In particular, suppose that (A, B) is stabilizable. Then the minimizer is given by $u_n = K x_n$ (i.e., state-feedback with constant gain), where

$$K = -(R + B'PB)^{-1} B'PA$$

and $P > 0$ is the unique solution to the discrete time algebraic Riaccati equation (DARE)

$$P = Q + A'(P - PB(R + B'PB)^{-1} B'P)A.$$

The value function of this optimization is $V(x_0) = x_0' P x_0$. Furthermore, $A + BK$ is stable. There is an alternative characterization of this P as the solution to the following LMI:

$$\begin{aligned} \max\ & \operatorname{tr}(P) \\ \text{s.t. }\ & P \geq 0 \\ & \begin{bmatrix} A'PA + Q - P & B'PA \\ A'PB & R + B'PB \end{bmatrix} \geq 0. \end{aligned}$$