IEOR 290A – Lecture 14
Stability

1 Nonlinear Dynamical Systems

Consider the following nonlinear dynamical system in discrete time:

$$x_{n+1} = f(x_n, u_n), \quad x_0 = \xi$$

where $x_n, x_{n+1} \in \mathbb{R}^p$, $u_n \in \mathbb{R}^q$, $f : \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}$ is a nonlinear function, the subscripts denote discrete time indices, and $\xi \in \mathbb{R}^p$ is an initial condition. In this model, the x_n are called *states*, and the u_n are *inputs*. The states represent the internal memory of the system, and they also summarize the history (past states and inputs) of the system. The inputs can be used to steer the system to satisfy engineering objectives.

A point x^* is an *equilibrium point* of the nonlinear system if there exists u^* such that $x^* = f(x^*, u^*)$. The intuition is that this is a point such that a constant input of u^* keeps the system at x^* whenever the system is already at x^*. Without loss of generality, we will assume that $x^* = 0$ and $u^* = 0$; note that this also implies that $f(0, 0) = 0$. (We can always define a translation of our coordinate system in which this assumption is true.)

2 Definitions of Stability

Such nonlinear systems have many interesting properties, and we will begin by discussing one type of property known as stability. In fact, there are several notions of stability in nonlinear systems, and the intuition is that a stable system has states that remain bounded.

2.1 Lyapunov Stability

A system is Lyapunov stable if given $M_2 > 0$ there exists $M_1 > 0$ such that $\|x_0\| \leq M_1$ implies that $\|x_n\| \leq M_2$ for all $n \geq 0$.

2.2 Asymptotic Stability

We say that a system is locally asymptotically stable (LAS) if (a) it is Lyapunov stable, and (b) there exists $M_3 > 0$ such that $\|x_n\| \to 0$ whenever $\|x_0\| \leq M_3$. A system is globally asymptotically stable (GAS) if $M_3 = \infty$.

2.3 Exponential Stability

A system is exponentially stable if (a) it is asymptotically stable, and (b) there exists $M_3 > 0$ and $\alpha, \beta > 0$ such that $\|x_n\| \leq \alpha\|x_0\| \exp(-\beta n)$ whenever $\|x_0\| \leq M_3$. A system is globally exponentially stable if $M_3 = \infty$.

3 Linear Dynamical Systems

An important special case of a nonlinear dynamical system is the situation in which the dynamics are linear and time invariant. In this case, the linear time invariance (LTI) dynamical system in discrete time can be described by

$$x_{n+1} = Ax_n + Bu_n, \quad x_0 = \xi$$

where $A \in \mathbb{R}^{p\times p}$ and $B \in \mathbb{R}^{p\times q}$. This special case is useful because it can be analyzed using powerful machinery from linear algebra.

The first thing to observe is that we can identify every equilibrium point of this linear system. Note that equilibrium points satisfy

$$x^* = Ax^* + Bu^* \Rightarrow \begin{bmatrix}\mathbb{I} - A & B\end{bmatrix}\begin{bmatrix}x^* \\ -u^*\end{bmatrix} = 0.$$

Thus, the set of all equilibrium points and the corresponding control inputs is characterized by the null space of the matrix $\begin{bmatrix}\mathbb{I} - A & B\end{bmatrix}$.

Stability is relatively straightforward for LTI systems. It turns out that LAS, GAS, and exponential stability are equivalent for LTI systems. Furthermore, LAS/GAS/exponential stability imply Lyapunov stability. Given a matrix A, there are multiple ways to check for exponential stability. An LTI system is exponentially stable if and only if

1. $|\sigma(A)| < 1$ (all eigenvalues have magnitude strictly less than one);
2. rank$(\begin{bmatrix}s\mathbb{I} - A\end{bmatrix}) = p$ for all $s \in \mathbb{C} : |s| \geq 1$;
3. given any $Q > 0$, there exists unique $P > 0$ such that $A'PA - P < -Q$; this equation is known as a Lyapunov equation, and it is a Linear Matrix Inequality (LMI) which means that it can be represented as a convex feasibility problem; the interpretation is that $x'Px$ is an energy function that decreases at rate

$$x'_{n+1}Px_{n+1} - x'_nPx_n = x'_nA'PAx_n - x'_nPx_n = x'_n(A'PA - P)x_n < -x'_nQx_n;$$

3.1 Bounded-Input Bounded-Output Stability

An LTI system is bounded-input bounded-output (BIBO) stable if there exists $0 \leq k < \infty$ such that for all bounded input sequences $\{u_0, u_1, \ldots\}$ the following holds:

$$\max_{n\geq 0}\|x_n\| \leq k \cdot \max_{n\geq 0}\|u_n\|.$$

One can think of the k as a gain. It turns out that BIBO stability is equivalent to LAS/GAS/exponential stability for LTI systems.