IEOR 151 – Service Operations Design and Analysis

Review Homework

12/03/2016

1. **Service Staffing Problem**
   Formulate the following service staffing problem as a linear integer program. Do not solve!
   You are hired by a bank to plan the workforce for their service operations. The planning horizon is 6 months from today. They currently have 12 tellers. The workforce hours required for the next 6 months are 1500, 1800, 1600, 2000, 1800, and 2200. You have the option of hiring trainees at the beginning of each month. However, each trainees should be hired a month before they can start working since they need one month training period. The training requires 80 hours of workforce hour from a regular teller for each trainee. At the end of each month, 10% of the workforce and trainees would quit. The direct labor cost is \$600 per month per teller, and \$300 per month per trainee. (A trainee would become a regular teller after the one month training period).
   Formulate the ILP which can be solved to generate the optimal workforce plan (i.e. the number of trainees you hire each month and the number of available of tellers in each month)
   Variables:

   - T_t - the number of trainees hired at the beginning of month t for $t = 1, \dots, 6$
   - A_t - the number of tellers available at the beginning of month t for $t = 1, \dots, 6$

   Assume the trainees cannot quit like regular tellers.
   answer:

$$
\begin{aligned}
\min \quad & \sum_{t=1}^{6} 600 A_t + \sum_{t=1}^{6} 300 T_t && (1)\\
\text{s.t.} \quad & A_1 = 12 && (2)\\
& A_2 = 0.9(T_1 + A_1) && (3)\\
& A_3 = 0.9(T_2 + A_2) && (4)\\
& A_4 = 0.9(T_3 + A_3) && (5)\\
& A_5 = 0.9(T_4 + A_4) && (6)\\
& A_6 = 0.9(T_5 + A_5) && (7)\\
& 160A_1 - 80T_1 \geq 1500 && (8)\\
& 160A_2 - 80T_2 \geq 1800 && (9)\\
& 160A_3 - 80T_3 \geq 1600 && (10)\\
& 160A_4 - 80T_4 \geq 2000 && (11)\\
& 160A_5 - 80T_5 \geq 1800 && (12)\\
& 160A_6 - 80T_6 \geq 2200 && (13)\\
& A_i, T_i \geq 0 \forall i && (14)
\end{aligned}
$$

2. **Service Staffing Problem II** Consider the previous problem again. However, consider the case where the bank can also receive temporary help from branches by offering temporary teller jobs to tellers under utilized in nearby branches. Due to the additional transportation cost needed, the temporary tellers are paid at \$900 per month. The number of available temporary tellers (B_t) at time t are 10, 7,

11, 3, 2, 6. Reformulate the problem as an ILP.
answer:
Introducing new decision variables in $Z_t \forall t = 1, \ldots, 15$ - the number of temporary tellers hired at the beginning of the month

$$\begin{aligned}
\min \quad & \sum_{t=1}^{6} 600A_t + \sum_{t=1}^{6} 300T_t + \sum_{t=1}^{6} 900Z_t && (15)\\
\text{s.t.} \quad & A_1 = 12 && (16)\\
& A_2 = 0.9(T_1 + A_1) && (17)\\
& A_3 = 0.9(T_2 + A_2) && (18)\\
& A_4 = 0.9(T_3 + A_3) && (19)\\
& A_5 = 0.9(T_4 + A_4) && (20)\\
& A_6 = 0.9(T_5 + A_5) && (21)\\
& 160A_1 - 80T_1 + 160Z_t \geq 1500 && (22)\\
& 160A_2 - 80T_2 + 160Z_t geq 1800 && (23)\\
& 160A_3 - 80T_3 + 160Z_t \geq 1600 && (24)\\
& 160A_4 - 80T_4 + 160Z_t \geq 2000 && (25)\\
& 160A_5 - 80T_5 + 160Z_t \geq 1800 && (26)\\
& 160A_6 - 80T_6 + 160Z_t \geq 2200 && (27)\\
& Z_t \leq B_t \quad \forall t && (28)\\
& A_t, T_t, Z_t \geq 0 \quad \forall t && (29)
\end{aligned}$$

3. **Savings Algorithm**Consider the nodes described below, and note that the depot is located at node 0. Suppose we would like to solve this vehicle routing problem (VRP) using the savings algorithm:

Distance	Node 0	Node 1	Node 2	Node 3	Node 4	Node 5	Node 6	Node 7	Node 8	Demand
Node 0		26	15	20	7	25	16	24	29	0
Node 1			15	23	26	33	40	38	54	18
Node 2				24	13	20	27	35	43	26
Node 3					26	42	34	15	39	11
Node 4						18	14	31	32	30
Node 5							25	49	45	21
Node 6								32	20	16
Node 7									30	29
Node 8										37

- Solve for the constraint that each vehicle has a capacity of 200
- Solve for the constraint that each vehicle has a capacity of 100
- Which solution yielded a better solution? Answer:
  First, write out the savings:

Savings: (sij = sji because the cost is symmetric)

$$s_{12} = 26 = 26 + 15 - 15 \ (= c_{01} + c_{02} - c_{12})$$
$$s_{13} = 23 = 26 + 20 - 23 \ (= c_{01} + c_{03} - c_{13})$$
$$s_{14} = 7 = 26 + 7 - 26 \ (= c_{01} + c_{04} - c_{14})$$
$$s_{15} = 18 = 26 + 25 - 33 \ (= c_{01} + c05 - c15)$$
$$s_{16} = 2 = 26 + 16 - 40 \ (= c_{01} + c_{06} - c_{16})$$
$$s_{17} = 12 = 26 + 24 - 38 \ (= c_{10} + c_{07} - c_{17})$$
$$s_{18} = 1 = 26 + 29 - 54 \ (= c_{01} + c_{08} - c_{18})$$
$$s_{23} = 11 = 15 + 20 - 24 \ (= c_{20} + c_{03} - c_{23})$$
$$s_{24} = 9 = 15 + 7 - 13 \ (= c_{20} + c_{04} - c_{24})$$
$$s_{25} = 20 = 15 + 25 - 20 \ (= c_{20} + c_{05} - c_{24})$$
$$s_{26} = 4 = 15 + 16 - 27 \ (= c_{20} + c_{06} - c_{24})$$
$$s_{27} = 4 = 15 + 24 - 35 \ (= c_{20} + c_{07} - c_{24})$$
$$s_{28} = 1 = 15 + 29 - 43 \ (= c_{20} + c_{08} - c_{28})$$
$$s_{34} = 1 = 20 + 7 - 26 \ (= c_{30} + c_{04} - c_{34})$$
$$s_{35} = 3 = 20 + 25 - 42 \ (= c_{30} + c_{05} - c_{35})$$
$$s_{36} = 2 = 20 + 16 - 34 \ (= c_{30} + c_{06} - c_{36})$$
$$s_{37} = 29 = 20 + 24 - 15 \ (= c_{30} + c_{07} - c_{37})$$
$$s_{38} = 10 = 20 + 29 - 39 \ (= c_{30} + c_{08} - c_{38})$$
$$s_{45} = 14 = 7 + 25 - 18 \ (= c_{40} + c_{05} - c_{45})$$
$$s_{46} = 9 = 7 + 16 - 14 \ (= c_{40} + c_{06} - c_{46})$$
$$s_{47} = 0 = 7 + 24 - 31 \ (= c_{40} + c_{07} - c_{47})$$
$$s_{48} = 4 = 7 + 29 - 32 \ (= c_{04} + c_{08} - c_{48})$$
$$s_{56} = 16 = 25 + 16 - 25 \ (= c_{50} + c_{06} - c_{56})$$
$$s_{57} = 0 = 25 + 24 - 49 \ (= c_{50} + c_{07} - c_{57})$$
$$s_{58} = 9 = 25 + 29 - 45 \ (= c_{50} + c_{08} - c_{58})$$
$$s_{67} = 8 = 16 + 24 - 32 \ (= c_{60} + c_{07} - c_{67})$$
$$s_{68} = 25 = 16 + 29 - 20 \ (= c_{60} + c_{08} - c_{68})$$
$$s_{78} = 23 = 24 + 29 - 30 \ (= c_{70} + c_{08} - c_{78})$$

Then, order the savings from largest to smallest:

$$
\begin{aligned}
s_{37} \\
s_{12} \\
s_{68} \\
s_{13}, s_{78}(tie) \\
s_{25} \\
s_{15} \\
s_{56} \\
s_{45} \\
s_{17} \\
s_{23} \\
s_{38} \\
s_{24}, s_{58}, s_{49}(tie) \\
\text{etc.}
\end{aligned}
$$

Applying the Savings algorithm:
For part a), vehicle capacity = 200. In this case, regardless of how you broke the ties, the algorithm will lead to the same solution:
Solution 1: At first tie, use s13 before s78

$$
\begin{aligned}
&s_{37} 0-3-7-0 \\
&s_{12} 0-3-7-0, 0-1-2-0 \\
&s_{68} 0-3-7-0, 0-1-2-0, 0-6-8-0 \\
&s_{13} 0-2-1-3-7-0, 0-6-8-0 \\
&s_{78} 0-2-1-3-7-8-6-0 \\
&s_{25} 0-5-2-1-3-7-8-6-0 \\
&s_{15} \quad x \\
&s_{56} \quad x \\
&s_{45} 0-4-5-2-1-3-7-8-6-0
\end{aligned}
$$

Solution 2: at first tie use s78 before s13:

$$
\begin{aligned}
&s_{37} 0-3-7-0 \\
&s_{12} 0-3-7-0, 0-1-2-0 \\
&s_{68} 0-3-7-0, 0-1-2-0, 0-6-8-0 \\
&s_{78} 0-6-8-7-3-0, 0-1-2-0 \\
&s_{13} 0-6-8-7-3-1-2-0 \\
&\text{same as with s13 first, so solution is same as above}
\end{aligned}
$$

Finally, calculate the cost: $Cost = 7 + 18 + 20 + 15 + 23 + 15 + 30 + 20 + 16 = 164$
For part b), vehicle capacity = 100. In this case, we will at least need 2 vehicles and we must keep track of demand on each route. Finally, note that the savings algorithm will lead to different solutions in this case.
Applying the savings algorithm:

Solution 1: At first tie, use s13 before s78

$$s370 - 3 - 7 - 0d = 40$$
$$s120 - 3 - 7 - 0d = 40, 0 - 1 - 2 - 0d = 44$$
$$s680 - 3 - 7 - 0d = 40, 0 - 1 - 2 - 0d = 44, 0 - 6 - 8 - 0d = 53$$
$$s130 - 2 - 1 - 3 - 7 - 0d = 84, 0 - 6 - 8 - 0d = 53$$
$$s78 \quad x$$
$$s25 \quad x$$
$$s15 \quad x$$
$$s560 - 2 - 1 - 3 - 7 - 0d = 84, 0 - 5 - 6 - 8 - 0d = 74$$
$$s45, s17, s23 \quad x$$

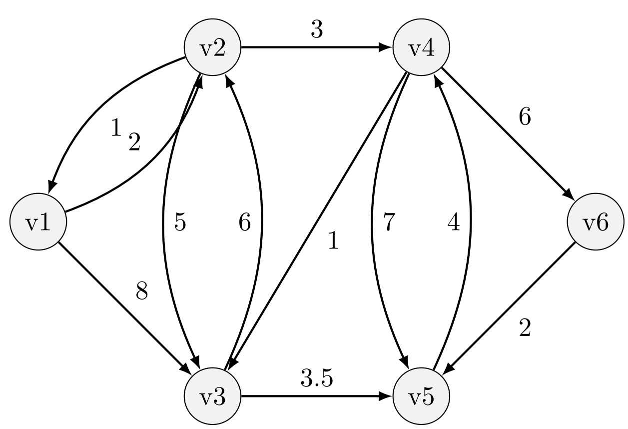
Note: only remaining customer is 4 with demand=30, so start a new route

$$0 - 2 - 1 - 3 - 7 - 0d = 84, 0 - 5 - 6 - 8 - 0d = 74, 0 - 4 - 0d = 30$$
$$Cost = 92 + 99 + 14 = 205$$

Solution 2: at first tie use s78 before s13

$$s370 - 3 - 7 - 0d = 40$$
$$s120 - 3 - 7 - 0d = 40, 0 - 1 - 2 - 0d = 44$$
$$s680 - 3 - 7 - 0d = 40, 0 - 1 - 2 - 0d = 44, 0 - 6 - 8 - 0d = 53$$
$$s780 - 3 - 7 - 8 - 6 - 0d = 93, 0 - 1 - 2 - 0d = 44$$
$$s13 \quad x$$
$$s250 - 3 - 7 - 8 - 6 - 0d = 93, 0 - 1 - 2 - 5 - 0d = 65$$
$$s15 \quad x$$
$$s56 \quad x$$
$$s450 - 3 - 7 - 8 - 6 - 0d = 93, 0 - 1 - 2 - 5 - 4 - 0d = 95$$
$$Cost = 101 + 86 = 187$$

4. **Deferred Acceptance Algorithm** Consider the following graph representation of a kidney exchange. Find the social welfare maximizaing exchange under the constraint that all cycles can have length less than or equal to L = 3. (5 points)



Solution:

Cycle Label	Cycles of $L \leq 3$	Cycle Weight	Disjoint Cycles	Weight
A	$v1 \to v2 \to v1$	3	A, G	15
B	$v2 \to v3 \to v2 \to v1$	11	A, F	11.5
C	$v4 \to v5 \to v4$	11	B, G	25
D	$v1 \to v3 \to v2 \to v1$	15	B, C	22
E	$v2 \to v4 \to v3 \to v2$	10	C, D	26
F	$v3 \to v5 \to v4 \to v3$	8.5	D, G	27
G	$v6 \to v5 \to v4 \to v6$	12	A, C	14

The Set of Disjoint Cycles {D, G} maximizes the social utility.

5. Consider Hummingbird Cafe on Euclid in which customers arrive at a rate of 3. The service times are exponentially distributed with mean of 0.08 hours, and there is currently only 1 checkout line.
Hint: The probability of a customer waiting in this $M/M/1$ queue is given by ρ^2 where $\rho = \frac{\lambda}{\mu}$, λ is the arrival rate, and μ is the service rate.
The answers were provided in the final review section and we repeat it here:

   (a) What is the average number of customers in the queue?
   Answer: The average number of customers in the queue is $L = \frac{\rho}{1-\rho}$, where $\rho = \frac{\lambda}{\mu} = 3*0.08 = 0.24$. Thus, $L = \frac{0.24}{0.76} = \frac{6}{19}$ customers

   (b) What is the average time spent in the system? What is the average time spent waiting in line (before service)?
   Answer: By Little's Law, we have that $L = \lambda W$, which implies that $\frac{6}{19} = 3 * W$. Solving this gives $W = \frac{2}{19}$ hours (or 6.31 minutes), which is the average time spent in the system. The average time spent waiting in line is $W_q = W - W_s =$ 6.31 - 60 * 0.08 = 1.51 mins

   (c) What is the probability of a customer having to wait?
   Answer: By the hint, the probability of waiting is $\rho^2 = 0.24^2 = 0.0576$
   You may also use the normal approximation: $1 - \Phi((1 - 0.24)/\sqrt{0.24}) = 0.061$

   (d) Suppose the boss would like the probability of a customer having to wait be below 15%. Use the square root law to determine the number of servers he should hire for an $M/M/s$ queue, in order to meet his objective?
   Answer: Using the squared root law, we should have

$$s = \rho + z_{0.85}\sqrt{\rho} = 0.24 + (1.04) * \sqrt{0.24} = 0.75$$

   where we have used the Normal probability table to find tht $Z_{0.85} = 1.04$. Thus, we should have $s = 1$ servers, since we must have an integral number of servers.

6. (Bonus) A quick proof of the hint in the previous question:
Let $p_k(t)$ be the probability that there are k customers in line at time t. In class, we showed that p_k is a geometric distribution (i.e. $p_k = (1-\rho)\rho^k$) and we will derive the result of the hint now.
The probability that a customer is waiting $= \sum_{k=2,3,\ldots} p_k = 1 - p_0 - p_1 = 1 - (1-\rho) - \rho(1-\rho) = 1 - 1 + \rho^2 = \rho^2$. Repeat the exercise to get a better understanding.