IEOR 151 – Service Operations Design and Analysis

Homework 4 (Due 12/02/2016)

11/18/2016

1. You are in charge of assigning classes to rooms and times at a small community college that offers evening classes from 6 to 10 in the evening. Some classes are scheduled MW from 6-8 and 8-10, while others are TTh from 6-8 and 8-10. In other words, there are 4 different time slots as shown below in Figure 1. There are therefore 12 room/time combinations. All of the information mentioned above are presented in Figure 1. In addition, you have enrollment information for the 11 courses that you will be offering. This information is given in the table in Figure 2. Each course has an ID number for ease of reference below as well.
   Clearly you do not want to assign a large enrollment class (e.g., US Politics with an enrollment of 55) to a small classroom (e.g., room 110 or 121). If you do, you incur a penalty that is 100 penalty units for each student above and beyond the capacity of the room; i.e., for each student you have to notify that he/she cannot take the course.
   While it is less serious, you also do not want to assign small classes (e.g., Advanced Excel) to a room with a lot of excess capacity. There is a penalty of 20 for each excess space in a room.
   Note: All related figures are presented on page 3 - 4
   Let us define the following notation:
   Inputs:
   - J: set of classes indexed by j
   - K: set of room/time combinations
   - c_u: Cost of undercapacity per student (100 penalty units)
   - c_o: Cost of overcapacity per student (20 penalty units)
   - h_j: enrollment in class j
   - s_k: space available in room/time l
   - p_{jk}: penalty for assigning class j to room/time k

   Decision Variables: Y_{jk}: 1 if class j is assigned to room/time k

   (a) Using the notation defined above, write down a formula or expression for the penalty p_{jk} in terms of the costs of undercapacity c_u and overcapacity c_o as well as the enrollment in the course h_j and the room capacity s_k.

   $$p_{jk} = \begin{cases} c_u(h_j - s_k) & \text{if } h_j - s_k \geq 0, \\ c_o(s_k - h_j) & h_j - s_k < 0 \end{cases}$$

   (b) Using the notation defined above, write down the objective of minimizing the total penalty of all assignments

   $$\min \sum_{k \in K} \sum_{j \in J} p_{jk} Y_{jk}$$

   (c) Using the notation defined above, write down the constraint that the number of classes assigned to a room/time combination cannot exceed 1

   $$\sum_{j \in J} Y_{jk} \leq 1 \quad \forall k \in K$$

(d) Using the notation defined above, write down the constraint that each class has to be assigned to a room/time

$$\sum_{k \in K} Y_{jk} = 1 \quad \forall j \in J$$

(e) With this formulation, do the variables Y_{jk} need to be binary, or can they simply be non-negative. Briefly justify your answer.
No, the do not since this is basically an assignment or transportation problem with all integer right-hand side values. Thus, the solution will be all integer and will automatically be 0/1 in this case.

(f) The solution to the problem is given below in Figure 3: The objective function (total penalty) is 1160. If you stare at this for some time, you will realize that the 2 EXCEL classes are scheduled on different days (Excel Basics on MW and Advanced Word on T Th). The same is true for the two WORD classes (Word Basics on MW and Advanced Word on T TH). The two Excel classes have the same instructor (Ms. Workbook) and the two Word classes have the same instructor (Mr. Paragraph). Both Ms. Workbook and Mr. Paragraph have said they want to teach on one night (either MW or T Th). (They do not have to on the same night, but both of of the Excel classes have to be on either MW or T Th and similarly both of the Word classes have to be on MW or T Th). Formulate linear constraints that will ensure that the schedule meets these requirements.

$$\sum_{k=1}^{6} Y_{2,k} + \sum_{k=7}^{12} Y_{7,k} \leq 1 \qquad \sum_{k=1}^{6} Y_{2,k} = \sum_{k=1}^{6} Y_{7,k}$$
$$\sum_{k=7}^{12} Y_{2,k} + \sum_{k=1}^{6} Y_{7,k} \leq 1 \qquad \sum_{k=7}^{12} Y_{2,k} = \sum_{k=7}^{12} Y_{7,k}$$
$$\sum_{k=1}^{6} Y_{3,k} + \sum_{k=7}^{12} Y_{11,k} \leq 1 \qquad \sum_{k=1}^{6} Y_{3,k} = \sum_{k=1}^{6} Y_{11,k}$$
$$\sum_{k=7}^{12} Y_{3,k} + \sum_{k=1}^{6} Y_{11,k} \leq 1 \qquad \sum_{k=7}^{12} Y_{3,k} = \sum_{k=7}^{12} Y_{11,k}$$

Note that these allow the teachers to teach the same class at the same time. To preclude this, we can use the following set of constraints:

$$\sum_{k=1}^{3} Y_{2,k} = \sum_{k=4}^{6} Y_{7,k} \text{ and } \sum_{k=4}^{6} Y_{2,k} = \sum_{k=1}^{3} Y_{7,k}$$
$$\sum_{k=7}^{9} Y_{2,k} = \sum_{k=10}^{12} Y_{7,k} \text{ and } \sum_{k=10}^{12} Y_{2,k} = \sum_{k=7}^{9} Y_{7,k}$$
$$\sum_{k=1}^{3} Y_{3,k} = \sum_{k=4}^{6} Y_{11,k} \text{ and } \sum_{k=4}^{6} Y_{3,k} = \sum_{k=1}^{3} Y_{11,k}$$
$$\sum_{k=7}^{9} Y_{3,k} = \sum_{k=10}^{12} Y_{11,k} \text{ and } \sum_{k=10}^{12} Y_{3,k} = \sum_{k=7}^{9} Y_{11,k}$$

(g) With these additional requirements, will the new total penalty be less than the old penalty, the same as the old penalty, or greater than the old penalty (or maybe something else). Again, briefly justify your answer.
The objective function will either stay the same or go up since you are adding a constraint to a minimization problem

2. Consider Stuffed Inn on Euclid in which customers arrive at a rate of 4. The service times are exponentially distributed with mean of 0.06 hours, and there is currently only 1 checkout line.

Hint: The probability of a customer waiting in this $M/M/1$ queue is given by $\rho = \frac{\lambda}{\mu}$, λ is the arrival rate, and μ is the service rate.

(a) What is the average number of customers in the queue?
Answer: The average number of customers in the queue is $L = \frac{\rho}{1-\rho}$, where $\rho = \frac{\lambda}{\mu} = 4*0.06 = 0.32$. Thus, $L = \frac{0.32}{01-0.32} = 0.316$ customers

(b) What is the average time spent in the system? What is the average time spent waiting in line (before service)?
Answers: By Little's Law, we have that $L = \lambda W$, which imples that $\frac{0.32}{4} = W$. Solving this gives $W = 0.02$ hours, which is the average time spent in the system. The average time spent waiting in line is $W_q = W - W_s = 0.08 - 0.06 = 0.02$ hours or 1.2 mins.

(c) What is the probability of a customer having to wait?
Answer: By the hint, the probability of waiting is $\rho^2 = 0.24^2 = 0.0576$

(d) Suppose the boss would like the probability of a customer having to wait be below 10%. Use the square root law to determine the number of servers he should hire for an $M/M/s$ queue, in order to meet his objective?
Answer: Using the squared root law, we should have

$$s = \rho + z_{0.9}\sqrt{\rho} = 0.24 + (1.29) * \sqrt{0.24} = 0.87,$$

where we have used the Normal probability table to find that $Z_{0.9} = 1.29$. Thus, we should have $s = 1$ servers, since we must have an integral number of servers.

3. Exercise 1 on page 504 in textbook chapter 9. Solve the problem using the modified parameters:

- customer arrival rate: 48 per hour
- mean service time: 3 mins
- For part b) the arrival rate for each server is 16 (or 1/3 of the system wide arrival rate)

Answer:
Arrival rate: $\lambda = 54/hr = 0.9/min$
Service rate: $\mu = \frac{1}{3}$

(a) Use $M/M/3$ queue with one line formula:

$$W_q = \frac{(\frac{\lambda}{s\mu})^{\sqrt{2(s+1)}-1}}{s(1-\frac{\lambda}{s\mu})} * \frac{1}{\mu} = 3.32mins$$

The waiting time in the queue is 3.32 minutes.

(b) Use $M/M/3$ queue with 3 lines formula:

$$W = \frac{1}{\mu - \lambda/3} = 15mins$$

$$W_q = W - W_s = 15 - 3 = 12mins$$

4. Exercise 11 on page 455 in textbook chapter 8. Feel free to use any solver at your disposal.
Answer:
Let $t = 1, \ldots, 15$, denote time where $t = 1$ is October in the previous year.
A_t: number of employees in month t
T_t: number of trainees hired in month t

D_t: number of needed employees in month t

$$
\begin{aligned}
\min \quad & 2500\sum_{t=4}^{12} A_t + 1500\sum_{t=1}^{12} T_t + 15000\sum_{t=1}^{12} X_t && (1)\\
s.t. \quad & T_{13} = 0 && (2)\\
& T_{14} = 0 && (3)\\
& T_{15} = 0 && (4)\\
& A_t = 0.9A_{t-1} + T_{t-3},\ t = 4, \dots, 15 && (5)\\
& A_3 = 100 && (6)\\
& A_t \geq D_t,\ \forall t && (7)\\
& 1000X_1 \geq T_1 && (8)\\
& 1000X_2 \geq T_2 && (9)\\
& 1000X_t \geq T_t + T_{t-1} + T_{t-2},\ t = 3, \dots, 12 && (10)\\
& X_t \in \{0, 1\}, \forall t = 1, \dots, 12 && (11)\\
& A_t \geq 0, \forall t && (12)\\
& T_t \geq 0, \forall t && (13)
\end{aligned}
$$

The code, in CPLEX, will be posted later.

5. Suppose we have $\frac{\lambda}{\mu} = 20$, and we would like to have the probability of waiting be at most 5%. Use the square root law to decide the number of services we should have.
Answer: Using the squared root law, we should have

$$s = \rho + z_{0.95}\sqrt{\rho} = 20 + (1.645) * \sqrt{20} = 27.4,$$

where we have used the Normal probability table to find that $Z_{0.95} = 1.645$. Thus, we should have $s = 27.4$ servers, since we must have an integral number of servers.

Days	Time	Room	Capacity	Room/Time Combination Number
M W	6-8	110	25	1
		121	45	2
		135	60	3
	8-10	110	25	4
		121	45	5
		135	60	6
T Th	6-8	110	25	7
		121	45	8
		135	60	9
	8-10	110	25	10
		121	45	11
		135	60	12

Figure 1: Class Schedule, Capacity, and Room/Time Combination Number

Course	Enrollment	Course ID #
Accounting	27	1
Advanced Excel	16	2
Advanced Word	24	3
Art Appreciation	52	4
Current Events	43	5
English Basics	55	6
Excel Basics	38	7
Nutrition	56	8
Personal Finance	42	9
US Politics	55	10
Word Basics	41	11

Figure 2: Course Enrollment Information

		Days	M W	M W	M W	M W	M W	M W	T Th	T Th	T Th	T Th	T Th	T Th
		Time	6-8	6-9	6-10	8-10	8-11	8-12	6-8	6-9	6-10	8-10	8-11	8-12
		Room	110	121	135	110	121	135	110	121	135	110	121	135
		Room/Time Combination Number	1	2	3	4	5	6	7	8	9	10	11	12
Course ID #	**Course**	**Enrollment**												
1	Accounting	27				**1**								
2	Advanced Excel	16							**1**					
3	Advanced Word	24										**1**		
4	Art Appreciation	52						**1**						
5	Current Events	43											**1**	
6	English Basics	55												**1**
7	Excel Basics	38					**1**							
8	Nutrition	56									**1**			
9	Personal Finance	42								**1**				
10	US Politics	55			**1**									
11	Word Basics	41		**1**										

Figure 3: Optimal Scheduling

Figure 3: Optimal Scheduling