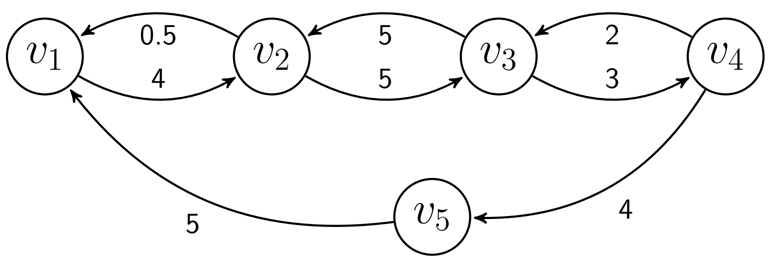
IEOR 151 – Lecture 8
Kidney Exchanges

1 Graph Model

We will consider a graph model for a kidney exchange[1]: There are k groups of donor-recipients (DR's), and the market is described by a directed graph $G = (V, E)$ with edge weights. There is a vertex $v_i \in V$ for each DR and an edge $e_{i,j} = (v_i, v_j) \in E$ from v_i to v_j if the i-th DR would accept a kidney from the j-th DR. The weight $w_{i,j}$ is the utility obtained to v_i of obtaining v_j's kidney. An example of a graph model for a kidney exchange market is shown below:



The clearing problem is the find the maximum weight union of disjoint cycles under the constraint that all cycles can have length less than or equal to a small constant L. It turns out that this problem is NP-hard, but it can be solved using integer programming.

2 Integer Linear Program (ILP) Solutions

2.1 Edge Formulation Solution

The first approach to solving this problem is to consider an optimization over edges. We start by defining the variable $s_{i,j}$ to be a variable that indicates whether edge $e_{i,j}$ is in some cycle. If $e_{i,j}$ is in some cycle, then $s_{i,j} = 1$; otherwise, $s_{i,j} = 0$ if edge $e_{i,j}$ is not in a cycle. With this definition, we can formulate the solution as the maximizer to the following integer

[1] D. Abraham, A. Blum, and T. Sandholm, "Clearing algorithms for barter exchange markets: Enabling nationwide kidney exchanges," Proceedings of the 8th ACM Conference on Electronic Commerce, pp. 295–304.

linear program (ILP):

$$\begin{aligned}
&\max \textstyle\sum_{s_{i,j},\forall e_{i,j}\in E} w_{i,j}s_{i,j} && \text{Sum Weight of Edges in cycles}\\
&\text{s.t. } s_{i,j}\in\{0,1\},\forall e_{i,j}\in E && \text{Variables are Binary}\\
&\quad \textstyle\sum_{e_{i,k}\in E} s_{i,k} - \sum_{e_{k,i}\in E} s_{k,i} = 0, \forall v_i \in V && \text{Conservation Constraint}\\
& && \text{(Outgoing minus Incoming Edges)}\\
&\quad \textstyle\sum_{e_{i,k}\in E} s_{i,k} \le 1, \forall v_i\in V && \text{Capacity Constraint}\\
& && \text{(Only one outgoing edge in a cycle)}\\
&\quad s_{i_1,i_2}+s_{i_2,i_3}+\ldots+s_{i_{L-1},i_L}\le L-1, && \text{Path Length Constraint}\\
&\qquad \forall L\text{-length paths}
\end{aligned}$$

Note that here a path is defined such that it cannot be a cycle (i.e., $i_1 \neq i_L$).

2.2 Cycle Formulation Solution

There is another approach to solving this problem. Let $C(L)$ be the set of all cycles of length L or less. We start by defining the variable t_c to be a variable that indicates whether cycle c is in the solution. If t_c is in the solution, then $t_c = 1$; otherwise, $t_c = 0$. With this definition, we can formulate the solution as the maximizer to the following ILP:

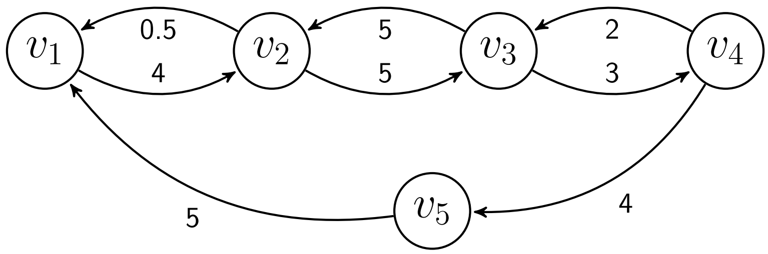
$$\begin{aligned}
&\max \textstyle\sum_{c\in C(L)} w_c t_c && \text{Sum Weight of Cycles}\\
&\text{s.t. } \textstyle\sum_{c:v_i\in c} t_c \le 1, \forall v_i\in V && \text{Each vertex in at most one cycle}\\
&\quad t_c\in\{0,1\},\forall c\in C(L) && \text{Variables are Binary}
\end{aligned}$$

2.3 Comparison of Formulations

There are some interesting facts about these two formulations. The edge formulation can be solved in polynomial time when there are no constraints on the maximum cycle size L, and the cycle formulation can be solved in polynomial time when the cycle size is at most 2. Also, the LP relaxation of the cycle formulation *weakly dominates* the LP relaxation of the edge formulation, meaning that the LP relaxation of the cycle formulation gives results that are in the worst case as good as those provided by the LP relaxation of the edge formulation. This is important because solving the ILP's for a kidney exchange at the nationwide scale is difficult because of memory and computational limitations, and so efficient algorithms make use of the LP relaxations. Lastly, for a graph with m edges, the edge formulation requires $O(m^3)$ constraints and the cycle formulation requires $O(m^2)$ constraints.

3 Example

To get a better understanding of this cycle formulation solution, we can solve it by hand. Recall the example from above:



First, we list all cycles of length $L \leq 3$ and compute the weight of these cycles. Next, we determine all sets of disjoint cycles and compute their weight. Lastly, the solution is the set of disjoint cycles with maximal weight. The steps are shown below, and the social welfare maximizing exchange is the set of disjoint cycles {B}.

Cycle Label	Cycles of $L \leq 3$	Cycle Weight	Disjoint Cycles	Weight
A	$v_1 \to v_2 \to v_1$	4.5	A, C	9.5
B	$v_2 \to v_3 \to v_2$	10	B	10
C	$v_3 \to v_4 \to v_3$	5		