IEOR 151 – Service Operations Design and Analysis

Homework 3 (Due 11/20/2015)

11/11/2015

1. In class, we looked at both the P-median and P-center model. The P-center model minimizes the maximum over all nodes distance between a demand node and the facility to which it is assigned, while the P-median model simply minimizes the demand weighted average distance. The constraints are the same for both problems. Therefore, we can use a **combined model** to find the tradeoff between the 2 objectives (minimizing the average and the maximum distances). Suppose we have two optimal solutions as shown in the table below.

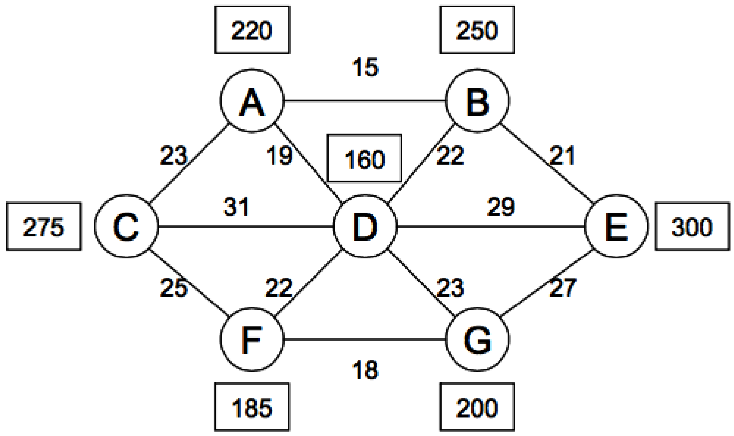
Solution	**Avg. Distance**	**Max. Distance**
1	150	500
2	200	425

   - Given the 2 optimal solutions, what should be the value of W for the weighted objective - W(Avg. Dist) + (1-W)(Max. Dist)?
     Answer:

$$\begin{aligned} 150W + 500(1-W) &= 200W + 425(1-W) \\ 75(1-W) &= 50W \\ W &= \frac{75}{125} = 0.6 \end{aligned}$$

   - Sketch the tradeoff curve.
     Answer: Simply connect the 2 points - (150, 500) & (200, 425)

2. Practitioners are often interested in achieving multiple objectives with varying priorities (different from question 1).
   Consider the following network:
   Numbers shown in square beside the nodes are the demand per week at the node.
   Consider the hierarchical objective problem of **first** finding the minimum number of facilities needed to cover all nodes within a coverage distance of 23, and then, from among all of the alternate optima for this problem, **second** finding the solution that maximizes the number of demands that are covered two or more times.



Define the following notation:
Inputs:

(a) J: Set of demand nodes
(b) K: Set of candidate nodes (same as the set of demand nodes in this case)
(c) a_{jk}: 1 if demand node j is covered by a facility located at node k; 0 otherwise
(d) h_j: demand at node j
(e) W: weight on the primary objective of minimizing the number of facilities

Decision Variables:

(a) X_k: 1 if facility is located at candidate site k; 0 if not
(b) Z_j: 1 if demand node j is covered 2 or more times; 0 if not

- Formulate the hierarchical objective of FIRST minimizing the number of facilities needed and SECOND maximizing the number of demands that are covered 2 or more times. Note that this should be a single function. This should be formulated in terms of notation.
  Answer: $\min W \sum_{k \in K} X_k - \sum_{j \in J} h_j Z_j$
- Formulate the constraints for this model including any integrality constraints. Explain the constraints using notations as well as words.
  Answer:

$$\text{Subject to:} \quad \begin{array}{ll} \sum_{k \in K} a_{jk} X_k - Z_j \geq 1 & \forall\, j \in J \\ X_k \in 0, 1 & \forall k \in K \\ Z_j \in 0, 1 & \forall j \in J \end{array}$$

- Do the Z_j variables have to be constrained to be binary, or can you simply use a constraint of the following form: $0 \leq Z_j \leq 1 \forall j$
  Answer: No, you do not. Once you require the location variables to be integer, the quantity $\sum_{k \in K} a_{jk} X_k$ will be integer and then the value of Z_j will be 0 if $\sum_{k \in K} a_{jk} X_k = 1$ and 1 if $\sum_{k \in K} a_{jk} X_k > 1$ since all demands are positive.
- How large should W be to ensure that the combined objective function of part (a) first minimizes the number of facilities and then selects the solution that maximizes the number of multiple covered demands from among the alternate optima? Briefly justify your answer. What you want is the smallest possible value of W.
  Answer: W should be greater than the sum of the demands so that adding an extra facility will hurt the first term of the objective function more than it can possibly benefit the second term

3. Consider the nodes described below, and note that the depot is located at node 0. Suppose we would like to solve this vehicle routing problem (VRP) using the savings algorithm:

Distance	Node 0	Node 1	Node 2	Node 3	Node 4	Node 5	Node 6	Node 7	Node 8	Demand
Node 0		26	15	20	7	25	16	24	29	0
Node 1			15	23	26	33	40	38	54	18
Node 2				24	13	20	27	35	43	26
Node 3					26	42	34	15	39	11
Node 4						18	14	31	32	30
Node 5							25	49	45	21
Node 6								32	20	16
Node 7									30	29
Node 8										37

- Solve for the constraint that each vehicle has a capacity of 200
- Solve for the constraint that each vehicle has a capacity of 100
- Which solution yielded a better solution?
  Answer:
  First, write out the savings:

Savings: (sij = sji because the cost is symmetric)

$$s12 = 26 = 26 + 15 - 15(c10 + c02 - c12 = c01 + c02 - c12)$$
$$s13 = 23 = 26 + 20 - 23(c10 + c03 - c13 = c01 + c03 - c13)$$
$$s14 = 7 = 26 + 7 - 26(c10 + c04 - c14 = c01 + c04 - c14)$$
$$s15 = 18 = 26 + 25 - 33(c10 + c05 - c15 = c01 + c05 - c15)$$
$$s16 = 2 = 26 + 16 - 40(c10 + c06 - c16 = c01 + c06 - c16)$$
$$s17 = 12 = 26 + 24 - 38(c10 + c07 - c17 = c01 + c07 - c17)$$
$$s18 = 1 = 26 + 29 - 54(c10 + c08 - c18 = c01 + c08 - c18)$$
$$s23 = 11 = 15 + 20 - 24(c20 + c03 - c23 = c02 + c03 - c23)$$
$$s24 = 9 = 15 + 7 - 13(c20 + c04 - c24 = c02 + c04 - c24)$$
$$s25 = 20 = 15 + 25 - 20(c20 + c05 - c24 = c02 + c05 - c25)$$
$$s26 = 4 = 15 + 16 - 27(c20 + c06 - c24 = c02 + c06 - c26)$$
$$s27 = 4 = 15 + 24 - 35(c20 + c07 - c24 = c02 + c07 - c27)$$
$$s28 = 1 = 15 + 29 - 43(c20 + c08 - c28 = c02 + c08 - c28)$$
$$s34 = 1 = 20 + 7 - 26(c30 + c04 - c34 = c03 + c04 - c34)$$
$$s35 = 3 = 20 + 25 - 42(c30 + c05 - c35 = c03 + c05 - c35)$$
$$s36 = 2 = 20 + 16 - 34(c30 + c06 - c36 = c03 + c06 - c36)$$
$$s37 = 29 = 20 + 24 - 15(c30 + c07 - c37 = c03 + c07 - c37)$$
$$s38 = 10 = 20 + 29 - 39(c30 + c08 - c38 = c03 + c08 - c38)$$
$$s45 = 14 = 7 + 25 - 18(c40 + c05 - c45 = c04 + c05 - c45)$$
$$s46 = 9 = 7 + 16 - 14(c40 + c06 - c46 = c04 + c06 - c46)$$
$$s47 = 0 = 7 + 24 - 31(c40 + c07 - c47 = c04 + c07 - c47)$$
$$s48 = 4 = 7 + 29 - 32(c04 + c08 - c48 = c04 + c08 - c48)$$
$$s56 = 16 = 25 + 16 - 25(c50 + c06 - c56 = c05 + c06 - c56)$$
$$s57 = 0 = 25 + 24 - 49(c50 + c07 - c57 = c05 + c07 - c57)$$
$$s58 = 9 = 25 + 29 - 45(c50 + c08 - c58 = c05 + c08 - c58)$$
$$s67 = 8 = 16 + 24 - 32(c60 + c07 - c67 = c06 + c07 - c67)$$
$$s68 = 25 = 16 + 29 - 20(c60 + c08 - c68 = c06 + c08 - c68)$$
$$s78 = 23 = 24 + 29 - 30(c70 + c08 - c78 = c07 + c08 - c78)$$

Then, order the savings from largest to smallest:

$$
\begin{aligned}
&s37\\
&s12\\
&s68\\
&s13, s78(tie)\\
&s25\\
&s15\\
&s56\\
&s45\\
&s17\\
&s23\\
&s38\\
&s24, s58, s49(tie)\\
&\text{etc.}
\end{aligned}
$$

Applying the Savings algorithm:
For part a), vehicle capacity = 200. In this case, regardless of how you broke the ties, the algorithm will lead to the same solution:
Solution 1: At first tie, use s13 before s78

$$
\begin{aligned}
&s370-3-7-0\\
&s120-3-7-0,0-1-2-0\\
&s680-3-7-0,0-1-2-0,0-6-8-0\\
&s130-2-1-3-7-0,0-6-8-0\\
&s780-2-1-3-7-8-6-0\\
&s250-5-2-1-3-7-8-6-0\\
&s15\quad x\\
&s56\quad x\\
&s450-4-5-2-1-3-7-8-6-0
\end{aligned}
$$

Solution 2: at first tie use s78 before s13:

$$
\begin{aligned}
&s370-3-7-0\\
&s120-3-7-0,0-1-2-0\\
&s680-3-7-0,0-1-2-0,0-6-8-0\\
&s780-6-8-7-3-0,0-1-2-0\\
&s130-6-8-7-3-1-2-0\\
&\text{same as with s13 first, so solution is same as above}
\end{aligned}
$$

Finally, calculate the cost: $Cost = 7 + 18 + 20 + 15 + 23 + 15 + 30 + 20 + 16 = 164$
For part b), vehicle capacity = 100. In this case, we will at least need 2 vehicles and we must keep track of demand on each route. Finally, note that the savings algorithm will lead to different solutions in this case.
Applying the savings algorithm:

Solution 1: At first tie, use s13 before s78

$s370 - 3 - 7 - 0d = 40$

$s120 - 3 - 7 - 0d = 40, 0 - 1 - 2 - 0d = 44$

$s680 - 3 - 7 - 0d = 40, 0 - 1 - 2 - 0d = 44, 0 - 6 - 8 - 0d = 53$

$s130 - 2 - 1 - 3 - 7 - 0d = 84, 0 - 6 - 8 - 0d = 53$

$s78 \quad x$

$s25 \quad x$

$s15 \quad x$

$s560 - 2 - 1 - 3 - 7 - 0d = 84, 0 - 5 - 6 - 8 - 0d = 74$

$s45, s17, s23 \quad x$

Note: only remaining customer is 4 with demand=30, so start a new route

$0 - 2 - 1 - 3 - 7 - 0d = 84, 0 - 5 - 6 - 8 - 0d = 74, 0 - 4 - 0d = 30$

$Cost = 92 + 99 + 14 = 205$

Solution 2: at first tie use s78 before s13

$s370 - 3 - 7 - 0d = 40$

$s120 - 3 - 7 - 0d = 40, 0 - 1 - 2 - 0d = 44$

$s680 - 3 - 7 - 0d = 40, 0 - 1 - 2 - 0d = 44, 0 - 6 - 8 - 0d = 53$

$s780 - 3 - 7 - 8 - 6 - 0d = 93, 0 - 1 - 2 - 0d = 44$

$s13 \quad x$

$s250 - 3 - 7 - 8 - 6 - 0d = 93, 0 - 1 - 2 - 5 - 0d = 65$

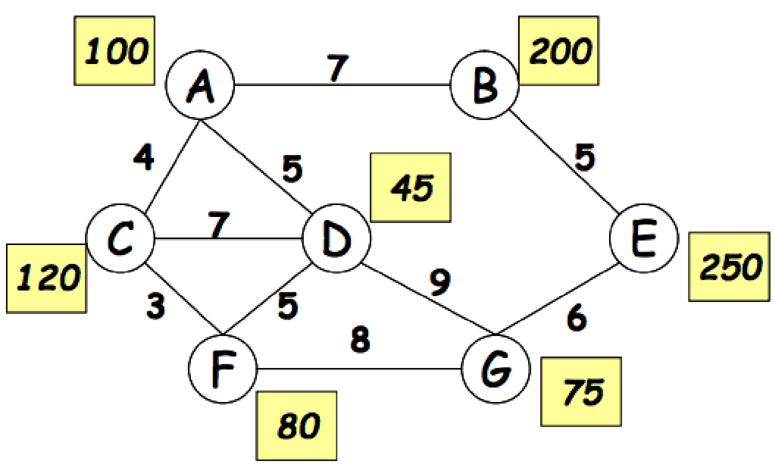
$s15 \quad x$

$s56 \quad x$

$s450 - 3 - 7 - 8 - 6 - 0d = 93, 0 - 1 - 2 - 5 - 4 - 0d = 95$

$Cost = 101 + 86 = 187$

4. Solve a P-median algorithm with the heuristic algorithm: allocate 2 facilities among 7 demand nodes (demand nodes set and candidate sites set are the same). The demand and distance information is given in the figure below.



Answer:

(a) Solve for the 1-median problem. Locate 1st facility at B, with total travel distance 5575.

(b) Fix 1st facility at B; Compute total travel distance with 2nd facility opened at A, C, D, $\ldots$, G. Locate 2nd facility at C. With B, C opened, total travel distance is 3030.

(c) Assign neighbors for B and C: {A, C, D, F, G} $\rightarrow$ C and {E, B} $\rightarrow$ E.

(d) Solve 1-median problem in each neighbor:

   i. In {A, C, D, F, G}, locate facility at C

   ii. In {E, B}, locate facility at E since E has larger demand

(e) Check whether there are any changes in the neighborhood and we realize that G is reassigned to E.

(f) Rerun Step 2 and we recognize that the termination condition is met and the final solution is: Location facilities at C and E with {A, C, D, F} $\rightarrow$ C and {B, E, G} $\rightarrow$ E.

5. For the same graph in question 5, solve a set-covering problem with the heuristic algorithm: cover all demands with covering distance of 10.
Answer:

(a) Let $p = 1$, solve for p-center problem. The longest demand-facility distance assigned= 14 $>$ covering distance of 10. That means, some demand can not be covered with $p = 1$.

(b) Let $p = 2$, solve for p-center problem. The longest demand-facility distance assigned= 9 $<$covering distance of 10. That means, now all demands can be covered.

(c) The minimum number of facilities needed to cover all demands is 2.