
IEOR 151 – Lecture 5

News vendor Model

1 Classical News vendor Model

Suppose we would like to pick the inventory level for a perishable good, where the demand is stochastic and the costs are deterministic. Here, the demand X is a random variable drawn from a distribution with fully known pdf $f(u)$ and cdf $F(u)$. Our decision space $\delta(u) \geq 0$, meaning that our inventory level must be non-negative. There are two possible cost structures for the classical news vendor model:

1.1 NO PRODUCTION COSTS

In one cost structure, the only costs are either from not having enough inventory to meet demand or loss from holding excess inventory that must be discarded because of the perishable nature of the good. We can write this cost structure using the following loss function

$$L(X, \delta) = -p \cdot (\delta - X)^- + q \cdot (\delta - X)^+,$$

where $p, q > 0$ and

$$(u)^- = \begin{cases} u, & \text{if } u \leq 0 \\ 0, & \text{otherwise} \end{cases} = \min\{u, 0\} = -\max\{-u, 0\}$$

and

$$(u)^+ = \begin{cases} u, & \text{if } u \geq 0 \\ 0, & \text{otherwise} \end{cases} = \max\{u, 0\}$$

The term $-p \cdot (\delta - X)^-$ represents loss from not being able to meet demand, and $q \cdot (\delta - X)^+$ represent loss from holding costs for having too much inventory for the actual demand.

If we would like to choose our inventory level to minimize the expected loss, then we would like to solve

$$\min_{\delta} \mathbb{E}(L(X, \delta)) = \min_{\delta} \int_{-\infty}^{\delta} q \cdot (\delta - x) f(x) dx + \int_{\delta}^{\infty} -p \cdot (\delta - x) f(x) dx.$$

If we use the Fundamental Theorem of Calculus to take the derivative of the function we are minimizing and set this derivative equal to zero, then we have

$$\begin{aligned}
 0 &= \int_{-\infty}^{\delta^*} q \cdot f(x)dx + \delta^* \cdot q \cdot f(\delta^*) - q \cdot \delta^* \cdot f(\delta^*) + \int_{\delta^*}^{\infty} -p \cdot f(x)dx + \delta^* \cdot p \cdot f(\delta^*) - p \cdot \delta^* \cdot f(\delta^*) \\
 &= \int_{-\infty}^{\delta^*} qf(x)dx - \int_{\delta^*}^{\infty} pf(x)dx \\
 &= qF(\delta^*) - p(1 - F(\delta^*))
 \end{aligned}$$

Solving for $F(\delta^*)$ in our stationarity condition $0 = qF(\delta^*) - p(1 - F(\delta^*))$, we get

$$F(\delta^*) = \frac{p}{p + q}.$$

Thus, the optimal inventory level is the $p/(p + q)$ -th quantile of the demand distribution.

As an example, suppose $X \sim \mathcal{N}(10, 20)$, $p = 1$, and $q = 4$, then the optimal inventory level is given by

$$F(\delta^*) = \Phi((\delta^* - 10)/\sqrt{20}) = \frac{1}{1 + 4} = \frac{1}{5} = 0.2.$$

From a z -table, we have that $\Phi(0.84) = 0.7995$, and so we would like to choose δ^* such that

$$(\delta^* - 10)/\sqrt{20} = -0.84 \Rightarrow \delta^* = 6.24.$$

1.2 PRODUCTION COSTS

A more general version of the newsvendor model includes a scenario with production costs. In this case, the loss function is given by

$$L(X, \delta) = c_f + c_v \cdot \delta - p \cdot (\delta - X)^- + q \cdot (\delta - X)^+,$$

where $p, q, c_f, c_v > 0$. Here, c_f is a fixed cost and c_v is a variable cost corresponding to the cost of producing an additional unit of perishable good. A similar calculation as above gives that the optimal inventory level is given by the δ^* that satisfies

$$F(\delta^*) = \frac{p - c_v}{p + q}.$$

Thus, the optimal inventory level is the $(p - c_v)/(p + q)$ -th quantile of the demand distribution.

As an example, suppose $X \sim \mathcal{N}(5, 10)$, $p = 5$, $q = 2$, and $c_v = 1$, then the optimal inventory level is given by

$$F(\delta^*) = \Phi((\delta^* - 5)/\sqrt{10}) = \frac{5 - 1}{5 + 2} = \frac{4}{7} = 0.5714.$$

From a z -table, we have that $\Phi(0.18) = 0.5714$, and so we would like to choose δ^* such that

$$(\delta^* - 5)/\sqrt{10} = 0.18 \Rightarrow \delta^* = 5.57.$$

1.3 PROFIT FORMULATION

There is an alternative way to formulate and solve the newsvendor problem. Instead of thinking about losses/costs, we can think of profits. The expected profit is given by

$$\mathbb{E}(r \cdot \min\{X, \delta\} - c_f - c_v \cdot \delta - q \cdot (\delta - X)^+)$$

where $r > 0$. Here, the interpretation is that we earn revenue of $r \cdot \min\{X, \delta\}$, and profit is given by revenue minus (i) fixed costs, (ii) variable cost, and (iii) holding cost. It turns out that maximizing expected profit leads to the same solution as the previous formulations of the newsvendor problem. To see why, note that maximizing $r \cdot \min\{X, \delta\} - c_f - c_v \cdot \delta - q \cdot (\delta - X)^+$ is the same as maximizing

$$\begin{aligned} & r \cdot \min\{X, \delta\} - c_f - c_v \cdot \delta - q \cdot (\delta - X)^+ - r \cdot X \\ &= r \cdot \min\{0, \delta - X\} - c_f - c_v \cdot \delta - q \cdot (\delta - X)^+ \\ &= r \cdot (\delta - X)^- - c_f - c_v \cdot \delta - q \cdot (\delta - X)^+ \\ &= -c_f - c_v \cdot \delta + r \cdot (\delta - X)^- - q \cdot (\delta - X)^+. \end{aligned}$$

Maximizing the expectation of this is equivalent to minimizing the expectation of its negative:

$$c_f + c_v \cdot \delta - r \cdot (\delta - X)^- + q \cdot (\delta - X)^+,$$

which is just the loss for the newsvendor problem, but with $p = r$.

There is another feature of this model regarding participation. It does not make sense to sell this perishable good unless the expected profit is positive. We will refer to the following inequality

$$\mathbb{E}(r \cdot \min\{X, \delta^*\} - c_f - c_v \cdot \delta^* - q \cdot (\delta^* - X)^+) \geq 0$$

as the *participation constraint* for the newsvendor model. Note that by the above argument, we have that the participation constraint can be written as

$$\mathbb{E}(rX - L(\delta^*, X)) \geq 0.$$

Recall that this δ^* satisfies $F(\delta^*) = \frac{r - c_v}{r + q}$.

2 Nonparametric Newsvendor

In many situations, we will not know what our distribution is. Instead, suppose we have iid samples of our demand X_1, \dots, X_n , which is drawn from some distribution with unknown cdf $F(u)$. Note that this iid assumption is quite strong, because we might instead have demand that is correlated in time. But for now, suppose we that this is not the case and that the demand is iid. A nonparametric formulation of the newsvendor problem is to compute δ^* to solve the following optimization problem

$$\min_{\delta} \frac{1}{n} \sum_{i=1}^n (c_f + c_v \cdot \delta - p \cdot (\delta - X_i)^- + q \cdot (\delta - X_i)^+).$$

The intuition is that this is the sample average of the loss. We can reformulate this as a linear program (LP)

$$\begin{aligned} \min \quad & \frac{1}{n} \sum_{i=1}^n (c_f + c_v \cdot \delta + p \cdot s_i + q \cdot t_i) \\ \text{s.t.} \quad & s_i \geq -(\delta - X_i) \\ & t_i \geq (\delta - X_i) \\ & s_i, t_i \geq 0. \end{aligned}$$

The key idea is that the two constraints $s_i \geq 0$ and $s_i \geq -(\delta - X_i)$ taken together are equivalent to the single constraint $s_i \geq \max\{0, -(\delta - X_i)\} = -\min\{0, (\delta - X_i)\} = -(\delta - X_i)^-$. Similarly, the two constraints $t_i \geq 0$ and $t_i \geq (\delta - X_i)$ taken together are equivalent to the single constraint $t_i \geq \max\{0, (\delta - X_i)\} = (\delta - X_i)^+$.