$\begin{array}{l} {\rm IEOR} \ 151-{\rm Midterm} \\ {\rm October} \ 22, \ 2014 \end{array}$

Name:	
Overall:	/48

Instructions:

- 1. Show all your intermediate steps.
- 2. You are allowed a single 8.5x11 inch note sheet.
- 3. Calculators are allowed.
- 4. "Normal probability table" is given on last page.

1	/10
2	/16
3	/10
4	/12

Note that the "Normal probability table" is from the LATEX source of: D. Diez, C. Barr, and M. Çetinkaya-Rundel, *OpenIntro Statistics*, 2012, under the Creative Commons BY-SA 3.0 license (http://creativecommons.org/licenses/by-sa/3.0/).

- 1. Suppose you are managing a restaurant, and you would like to determine if the restaurant should purchase a new electronic system to manage orders. Purchasing the system will cost \$4950, and other restaurants have found that using the electronic system leads to an average of 3 mistakes per day. If the current error rate is 5 mistakes per day, then purchasing the new system will have a net savings of \$2500 over the span of one year. You have decided to use a minmax hypothesis testing approach to answer this question. As a first step, you recorded the number of mistakes made over five days: 2, 2, 3, 7, 4.
 - (a) Assume that the number of mistakes per day is approximated by a Gaussian random variable with variance $\sigma^2 = 0.75$. Using a binary search and a z-table, compute the threshold for this hypothesis test γ^* to within an accuracy of ± 0.1 . (4 points)

Hint: Use the following values for the minmax hypothesis test: n = 5, $\mu_0 = 3$, $\mu_1 = 5$, $\sigma^2 = 0.75$, $L(\mu_0, d_0) = 0$, $L(\mu_0, d_1) = a = 4950$, $L(\mu_1, d_0) = b = 2500$, $L(\mu_1, d_1) = 0$.

- (b) Should the restaurant purchase the new electronic system? Explain your answer. (2 points)
- (c) Instead of the previous conditions, assume that the number of mistakes per day is a uniform random variable with variance 3. Should the restaurant purchase the new electronic system? Explain your answer. (4 points)

Hint: The hypotheses are H_0 : $\mathcal{U}[0,6]$ and H_1 : $\mathcal{U}[2,8]$, where $\mathcal{U}[a,b]$ denotes a uniform random variable with support from a to b.

Solutions:

(a) The goal is to find the γ^* that satisfies

$$4950 \cdot (1 - \Phi(\sqrt{5}(\gamma - 3)/\sqrt{0.75})) = 2500 \cdot \Phi(\sqrt{5}(\gamma - 5)/\sqrt{0.75}),$$

or equivalently

$$4950 \cdot (1 - \Phi(2.582 \cdot (\gamma - 3))) = 2500 \cdot \Phi(2.582 \cdot (\gamma - 5))$$

Since $L(\mu_0, d_1) > L(\mu_1, d_0)$, the threshold must lie in the range (4, 5). One possible first guess is the middle of this range: 4.5. The first guess and subsequent guesses of the binary search are summarized in the table below. The final solution is that $\gamma^* \in (4, 4.13)$, which is within accuracy of ± 0.07 .

Step	γ	LHS	RHS
1	4.5	0	246
2	4.25	3	66
3	4.13	9	31

- (b) The sample average is $\overline{X} = (2+2+3+7+4)/5 = 3.6$. This is less than γ^* , and so accept the hypothesis that the mean number of errors is $\mu_0 = 2$. Consequently, the restaurant would decide to NOT purchase the new electronic system under these set of assumptions.
- (c) The hypothesis $H_1 : \mathcal{U}[2,8]$ should be accepted, because the measurement on the fourth day of 7 errors is not consistent with the other hypothesis $H_0 : \mathcal{U}[0,6]$. As a result, the restaurant would decide to purchase the new electronic system under this alternative set of assumptions.

- 2. Suppose you are managing a bakery and would like to determine the number of croissants that should be baked in the morning. Assume the selling price is \$1.95 (r = 1.95), the per unit production cost is \$0.75 ($c_v = 0.75$), and the holding cost is \$0.25 (q = 0.25).
 - (a) Assume the demand is distributed as a Gaussian random variable with mean 150 and variance 5000. Use the newsvendor model to choose how many croissants should be produced? (4 points)
 - (b) Suppose you have measured the demand of croissants for the past 10 days, and the values of demand sorted into ascending order are 76, 93, 119, 125, 128, 132, 163, 187, 217, 285. Use the nonparametric newsvendor model to choose how many croissants the bakery should produce. Explain your reasoning. (4 points)

Hint: Consider the sample distribution

$$\hat{F}(z) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}(z \le X_i).$$

(c) Suppose that you also measure outside temperature T_i and rainfall R_i , which you believe are correlated with demand for croissants. Using these measurements, you model croissant demand as a linear function of outside temperature and rainfall: $X_i = a \cdot T_i + b \cdot R_i + c$, where a, b, c are unknown constants. Furthermore, you decide to use statistical regularization to improve the results by adding the constraint $|a| + |b| \leq \mu$, where μ is a known constant. The resulting newsvendor model with a linear demand model and L1-constraint regularization is written as

$$\min \frac{1}{n} \sum_{i=1}^{n} (c_f + c_v \cdot \delta_i - p \cdot (\delta_i - X_i)^- + q \cdot (\delta_i - X_i)^+)$$

s.t. $\delta_i = a \cdot T_i + b \cdot R_i + c, \ \forall i$
 $|a| + |b| \le \mu.$

Write this formulation as a linear program. (4 points)

Hint: Recall that the nonparametric newsvendor

$$\min \frac{1}{n} \sum_{i=1}^{n} (c_f + c_v \cdot \delta - p \cdot (\delta - X_i)^- + q \cdot (\delta - X_i)^+).$$

can be written as the following linear program

$$\min \frac{1}{n} \sum_{i=1}^{n} (c_f + c_v \cdot \delta + p \cdot s_i + q \cdot t_i)$$

s.t. $s_i \ge -(\delta - X_i), \ \forall i$
 $t_i \ge (\delta - X_i), \ \forall i$
 $s_i, t_i \ge 0, \ \forall i.$

(d) Suppose you would like to perform the statistical regularization in another way. Let $\lambda > 0$ be a known constant, and suppose you solve the following form of the newsvendor with linear demand and L1-regularization

$$\min \frac{1}{n} \sum_{i=1}^{n} (c_f + c_v \cdot \delta_i - p \cdot (\delta_i - X_i)^- + q \cdot (\delta_i - X_i)^+) + \lambda \cdot (|a| + |b|)$$

s.t. $\delta_i = a \cdot T_i + b \cdot R_i + c, \ \forall i.$

Write this formulation as a linear program. (4 points)

Solutions:

(a) The optimal inventory level δ^* is given by

$$F(\delta^*) = \Phi((\delta^* - 150) / \sqrt{5000}) = \frac{1.95 - 0.75}{1.95 + 0.25} = 0.5455.$$

From a z-table, it holds that $\Phi(0.11) = 0.5438$, and so choose δ^* such that

$$(\delta^* - 150)/\sqrt{5000} = 0.11 \Rightarrow \delta^* = 158.$$

(b) Observe that the nonparametric newsvendor model is equivalent to the newsvendor model applied to the sample distribution

$$\hat{F}(z) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}(z \le X_i).$$

As a result, the optimal inventory level δ^* is given by

$$\hat{F}(\delta^*) = \frac{1.95 - 0.75}{1.95 + 0.25} = 0.5455.$$

The 0.5455-quantile value is the sixth measured value 132, since $\hat{F}(132^{-}) = 0.5$ and $\hat{F}(132) = 0.6$. Summarizing, the nonparametric newsvendor model says the bakery should bake 132 croissants.

(c) By adding the auxiliary variables u, v, this problem can be written as the following linear program

$$\min \frac{1}{n} \sum_{i=1}^{n} (c_f + c_v \cdot \delta_i + p \cdot s_i + q \cdot t_i)$$

s.t. $s_i \ge -(\delta_i - X_i), \forall i$
 $t_i \ge (\delta_i - X_i), \forall i$
 $s_i, t_i \ge 0, \forall i$
 $\delta_i = a \cdot T_i + b \cdot R_i + c, \forall i$
 $-u \le a \le u$
 $-v \le b \le v$
 $u + v = \mu.$

(d) By adding the auxiliary variables u_i, v_i, z , this problem can be written as the following linear program

$$\min \frac{1}{n} \sum_{i=1}^{n} (c_f + c_v \cdot \delta_i + p \cdot s_i + q \cdot t_i) + \lambda \cdot z$$

s.t. $s_i \ge -(\delta_i - X_i), \ \forall i$
 $t_i \ge (\delta_i - X_i), \ \forall i$
 $s_i, t_i \ge 0, \ \forall i$
 $\delta_i = a \cdot T_i + b \cdot R_i + c, \ \forall i$
 $-u \le a \le u$
 $-v \le b \le v$
 $u + v \le z.$

3. Consider the following graph representation of a kidney exchange. Find the social welfare maximizing exchange under the constraint that all cycles can have length less than or equal to L = 3. (10 points)



Solutions:

First, list all cycles of length $L \leq 3$ and compute the weight of these cycles. Next, determine all sets of disjoint cycles and compute their weight. Lastly, the solution is the set of disjoint cycles with maximal weight. The steps are shown below, and the social welfare maximizing exchange is the cycle B.

Cycle Label	Cycles of $L \leq 3$	Cycle Weight	Disjoint Cycles	Weight
A	$v_1 \rightarrow v_2 \rightarrow v_1$	8	А	8
В	$v_1 \to v_2 \to v_4 \to v_1$	14	В	14
C	$v_1 \rightarrow v_3 \rightarrow v_1$	7	С	7
D	$v_1 \to v_3 \to v_4 \to v_1$	8	D	8

- 4. Suppose you are consulting for a hospital that would like to hire a contractor to wash hospital clothing (e.g., scrubs). The fixed costs for the contractor are \$1200, and the hospital's utility for washed clothing is given by $S(q) = 40\sqrt{q}$. If the contractor is inefficient then its marginal costs are \$0.30, and if the contractor is efficient then its marginal costs are \$0.25. Assume that the hospital believes that there is a 30% (3/10) chance that the contractor is efficient. Your boss wants to choose a menu of contracts such that the inefficient contractor does not participate. The boss suggests the menu of contracts $C_B = \{(q^I = 0, t^I = 0), (q^E = 6400, t^E = 2800)\}.$
 - (a) Verify that the boss's suggested menu of contracts C_B satisfies the participation constraint and the incentive compatibility constraint for the efficient agent. (2 points)
 - (b) Verify that the boss's suggested menu of contracts C_B does NOT satisfy the participation constraint for the inefficient agent. (2 points)
 - (c) What is the menu of contracts for the second-best production levels? (4 points)
 - (d) Which menu of contracts (the boss's suggestion OR the menu for second-best levels) is better? Explain your reasoning. (4 points)

Solutions:

- (a) The participation constraint for the efficient agent is $t^E \theta^E q^E F \ge 0$. Since $t^E \theta^E q^E F = 2800 0.25 \cdot 6400 1200 = 0$, this participation constraint is satisfied. The incentive compatibility constraint for the efficient agent is $t^E \theta^E q^E F \ge t^I \theta^E q^I F$. Since $t^E \theta^E q^E F = 2800 0.25 \cdot 6400 1200 = 0$ and $t^I \theta^E q^I F = 0 0.25 \cdot 0 1200 = -1200$, this incentive compatibility constraint is satisfied.
- (b) The participation constraint for the inefficient agent is $t^{I} \theta^{I}q^{I} F \ge 0$. Since $t^{I} \theta^{I}q^{I} F = 0 0.30 \cdot 0 1200 = -1200$, this participation constraint is NOT satisfied.
- (c) The production level for the efficient agent is

$$q_2^E : S'(q_2^E) = \theta^E \Rightarrow q_2^E : \frac{20}{\sqrt{q_2^E}} = 0.25 \Rightarrow q_2^E = 6400.$$

The production level for the inefficient agent is

$$q_2^I: S'(q_2^I) = \theta^I + \frac{\nu}{1-\nu} (\theta^I - \theta^E) \Rightarrow q_2^I: \frac{20}{\sqrt{q_2^I}} = 0.30 + \frac{0.3}{1-0.3} (0.30 - 0.25) \Rightarrow q_2^I = 3871$$

The transfer for the efficient agent is

$$t_2^E = \theta^E q_2^E + (\theta^I - \theta^E) q_2^I + F = 0.25 \cdot 6400 + (0.30 - 0.25) \cdot 3871 + 1200 = 2993.55,$$

and the transfer for the inefficient agent is

$$t_2^I = \theta^I q_2^I + F = 0.30 \cdot 3871 + 1200 = 2361.30.$$

Summarizing, the menu of contracts for the second-best production levels are $\{(q_2^E = 6400, t_2^E = 2993.55), (q_2^I = 3871, t_2^I = 2361.30)\}.$

(d) With the boss's menu of contracts, the expected utility minus transfers is

$$\nu(S(q^E) - t^E) + (1 - \nu)(S(q^I) - t^I)$$

= 0.3 \cdot (40\sqrt{6400} - 2800) + 0.7 \cdot (40\sqrt{0} - 0) = 120.

With the contract corresponding to the second-best level of production, the expected utility minus transfers is

$$\nu(S(q^E) - t^E) + (1 - \nu)(S(q^I) - t^I)$$

= 0.3 \cdot (40\sqrt{6400} - 2993.55) + 0.7 \cdot (40\sqrt{3871} - 2361.30) = 151.11.

Thus, the menu of contracts corresponding to the second-best production levels is better in this particular scenario.

Normal probability table



	Second decimal place of Z									
Z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

*For $Z \ge 3.50$, the probability is greater than or equal to 0.9997