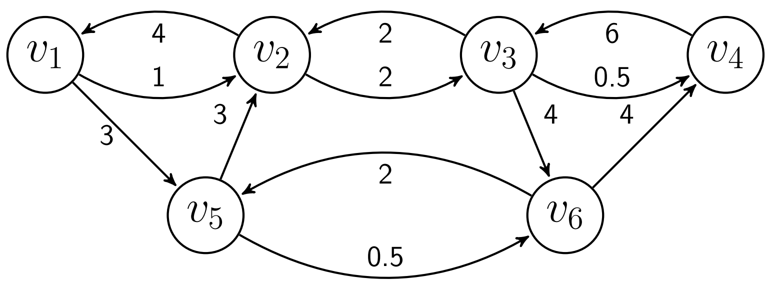
IEOR 151 – Homework 2
Due Friday, October 17, 2014 in class

1. Consider the following graph representation of a kidney exchange. Find the social welfare maximizing exchange under the constraint that all cycles can have length less than or equal to $L = 3$. (5 points)



First, we list all cycles of length $L \leq 3$ and compute the weight of these cycles. Next, we determine all sets of disjoint cycles and compute their weight. Lastly, the solution is the set of disjoint cycles with maximal weight. The steps are shown below, and the social welfare maximizing exchange is the set of disjoint cycles B, E.

Cycle Label	Cycles of $L \leq 3$	Cycle Weight	Disjoint Cycles	Weight
A	$v_1 \to v_2 \to v_1$	5	A, D, F	14
B	$v_1 \to v_5 \to v_2 \to v_1$	10	A, E	19
C	$v_2 \to v_3 \to v_2$	4	A, F	7.5
D	$v_3 \to v_4 \to v_3$	6.5	B, D	16.5
E	$v_3 \to v_6 \to v_4 \to v_3$	14	B, E	24
F	$v_5 \to v_6 \to v_5$	2.5	C, F	6.5
			D, F	9
			E	14
			F	2.5

2. Match the applicants to the residency programs, and show intermediate steps of the algorithm. (5 points)

   For this problem, suppose the applicants' preferences are given by:

Bob	**Louise**	**Lisa**
1. City	1. City	1. City
2. General	2. General	2. General
3. Mercy	3. Mercy	3. Mercy

   Suppose that each residency program has only 1 open position, and that the preferences of the programs are given by

City	**General**	**Mercy**
1. Lisa	1. Lisa	1. Lisa
2. Louise	2. Bob	2. Louise
3. Bob	3. Louise	3. Bob

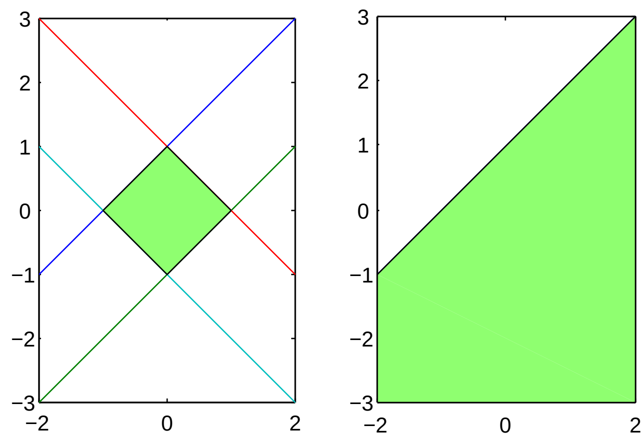
   The results are given by the following table.

City	**General**	**Mercy**
~~Bob~~	Bob	Louise
~~Louise~~		
Lisa		

3. Consider the following optimization problem denoted (P)

$$\begin{aligned} &\min\ x_1 - x_2 \\ &\text{s.t. } |x_1| + |x_2| \leq 1. \end{aligned}$$

   (a) The constraint $|x_1|+|x_2|-1 \leq 0$ is not differentiable, and so we cannot use the KKT conditions to solve this problem. This can be resolved by rewriting this constraint so that it is differentiable. Note that the corresponding feasible set is indicated by the shaded green square on the left side of the image below. One way to represent this feasible set is as the intersection of four half-spaces. For instance, the half-space $-x_1 + x_2 - 1 \leq 0$ is shown on the right side of the image below.



Write the constraint $|x_1| + |x_2| - 1 \leq 0$ as the intersection of four half-spaces. (2 points)

We can write this constraint as

$$\begin{aligned} &x_1 + x_2 - 1 \leq 0 \\ &x_1 - x_2 - 1 \leq 0 \\ &-x_1 + x_2 - 1 \leq 0 \\ &-x_1 - x_2 - 1 \leq 0. \end{aligned}$$

(b) Another representation of the constraint $|x_1| + |x_2| - 1 \leq 0$ is as the projection of a set. For example, we can write $|x_1| \leq 1$ as the projection of a polytope by introducing an additional variable t_1:

$$\begin{aligned} &-t_1 \leq x_1 \leq t_1 \\ &t_1 = 1 \end{aligned}$$

Write the constraint $|x_1|+|x_2|-1 \leq 0$ as the projection of a polytope by introducing two additional variables t_1, t_2. (2 points)

We can write this constraint as

$$\begin{aligned} &-t_1 \leq x_1 \leq t_1 \\ &-t_2 \leq x_2 \leq t_2 \\ &t_1 + t_2 = 1. \end{aligned}$$

(c) Suppose we had a constraint $\sum_{i=1}^{n} |x_i| - 1 \leq 0$, which is not differentiable. Write this constraint as the intersection of half-spaces. How many variables do we need? How many constraints/inequalities are there? (4 points)

We can write this constraint as the set of constraints:

$$\sum_{i=1}^{n} \pm x_i - 1 \leq 0,$$

where the $\pm$ indicates that every possible permutation of signs is included in this set of constraints. There are n variables $x_1, \ldots, x_n$ and 2^n constraints/inequalities.

(d) Suppose we had a constraint $\sum_{i=1}^{n} |x_i| - 1 \leq 0$, which is not differentiable. Write this constraint as the projection of a polytope. How many variables do we need? How many constraints/inequalities are there? (4 points)

We can write this constraint as

$$\begin{aligned} & -t_i \leq x_i \leq t_i, \forall i = 1, \ldots, n \\ & \sum_{i=1}^{n} t_i = 1. \end{aligned}$$

There are $2n$ variables $x_1, \ldots, x_n, t_1, \ldots, t_n$ and $2n + 1$ constraints/inequalities.

(e) Recall the original optimization problem (P). Reformulate the optimization problem (P) so that the constraints are differentiable, and then use the KKT conditions to solve the optimization problem. Hint: There are an infinite number of solutions to (P). (4 points)

One reformulation of (P) is

$$\begin{aligned} \min \ & x_1 - x_2 \\ \text{s.t.} \ & -t_1 - x_1 \leq 0 \\ & x_1 - t_1 \leq 0 \\ & -t_2 - x_2 \leq 0 \\ & x_2 - t_2 \leq 0 \\ & t_1 + t_2 = 1. \end{aligned}$$

The KKT conditions are

$$
\begin{aligned}
&1-\lambda_1+\lambda_2=0\\
&-1-\lambda_3+\lambda_4=0\\
&-\lambda_1-\lambda_2+\mu_1=0\\
&-\lambda_3-\lambda_4+\mu_1=0\\
&-t_1-x_1\le 0\\
&x_1-t_1\le 0\\
&-t_2-x_2\le 0\\
&x_2-t_2\le 0\\
&t_1+t_2=1\\
&\lambda_1,\lambda_2,\lambda_3,\lambda_4\ge 0\\
&\lambda_1(-t_1-x_1)=0\\
&\lambda_2(x_1-t_1)=0\\
&\lambda_3(-t_2-x_2)=0\\
&\lambda_4(x_2-t_2)=0.
\end{aligned}
$$

First, note that $\lambda_1 > 0$ (strictly greater than zero) because otherwise $1-\lambda_1+\lambda_2=0$ cannot be satisfied. A similar argument shows that $\lambda_4 > 0$ (strictly greater than zero). These two implications indicate that $-t_1-x_1=0$ and $x_2-t_2=0$. But since $t_1+t_2=1$, we have that

$$
\begin{aligned}
x_2 &= t_2\\
x_1 &= t_2-1.
\end{aligned}
$$

Note that $t_2 \ge 0$, since otherwise the inequalities $-t_2 \le x_2 \le t_2$ cannot be satisfied. Thus, the solutions to (P) are given by the set $\{(x_1,x_2) : x_1 = t_2-1 \text{ and } x_2 = t_2 \text{ and } 0 \le t_2 \le 1\}$.

4. Suppose a fast food restaurant would like to purchase veggie burger patties from a food distributor. The restaurant's utility for the patties is given by $S(q) = 500ln(1+q)$. The fixed costs for the food distributor are 4,000, and if the distributor is inefficient (efficient) then its marginal costs are 0.15 (0.10). Assume that the restaurant believes that there is a 30% chance that the food distributor is efficient.

   (a) What are the first-best production levels? (2 points)

   Note that equating marginal utility to marginal costs for the inefficient distributor gives

$$
\begin{aligned}
q_1^I : S'(q_1^I) = \theta^I &\Rightarrow q_1^I : \frac{500}{1+q_1^I} = 0.15\\
&\Rightarrow q_1^I = 3332.
\end{aligned}
$$

Similarly, equating marginal utility to marginal costs for the efficient distributor gives

$$q_1^E : S'(q_1^E) = \theta^E \Rightarrow q_1^E : \frac{500}{1+q_1^E} = 0.10$$
$$\Rightarrow q_1^E = 4999.$$

(b) What are the contracts to implement the first-best production levels? (2 points)

These contracts allow for zero information rent, meaning that the inefficient distributor should be offered the contract

$$(q_1^I = 3332, t_1^I = \theta^I q_1^I + F = 0.15 \cdot 3332 + 4000 = 4499.80),$$

and the efficient distributor should be offered the contract

$$(q_1^E = 4999, t_1^E = \theta^E q_1^E + F = 0.10 \cdot 4999 + 4000 = 4499.90).$$

(c) How much profit would the efficient distributor make if the restaurant offers a menu of contracts $\{(q_1^I, t_1^I), (q_1^E, t_1^E)\}$? (1 point)

The profit would be

$$t_1^I - \theta^E q_1^I - F = 4499.80 - 0.10 \cdot 3332 - 4000 = 166.60.$$

(d) What are the second-best production levels? (2 points)

The production level for the efficient agent remains unchanged $q_2^E = q_1^E = 4999$, and the production level for the inefficient agent decreases to

$$q_2^I : S'(q_2^I) = \theta^I + \frac{\nu}{1-\nu}(\theta^I - \theta^E) \Rightarrow q_2^I : \frac{500}{1+q_2^I} = 0.15 + \frac{0.3}{1-0.3}(0.15 - 0.10)$$
$$\Rightarrow q_2^I = 2916.$$

(e) What is the menu of contracts for the second-best production levels? (2 points)

The transfer for the efficient agent is

$$t_2^E = \theta^E q_2^E + (\theta^I - \theta^E) q_2^I + F = 0.10 \cdot 4999 + (0.15 - 0.10) \cdot 2916 + 4000 = 4645.70,$$

and the transfer for the inefficient agent is

$$t_2^I = \theta^I q_2^I + F = 0.15 \cdot 2916 + 4000 = 4437.40.$$

Summarizing, the menu of contracts are $\{(q_2^E = 4999, t_2^E = 4645.70), (q_2^I = 2916, t_2^I = 4437.40)\}$.

(f) What is the information rent of the efficient distributor for the menu of contracts for the second-best production levels? Is this higher or lower than the profit gained for the menu of contracts for the first-best production levels? (2 points)

The information rent for the efficient distributor is

$$U^E = t^E - \theta^E q^E - F = 4645.70 - 0.10 \cdot 4999 - 4000 = 145.80.$$

This is lower than the profit gained for the menu of contracts for the first-best production levels 166.60.