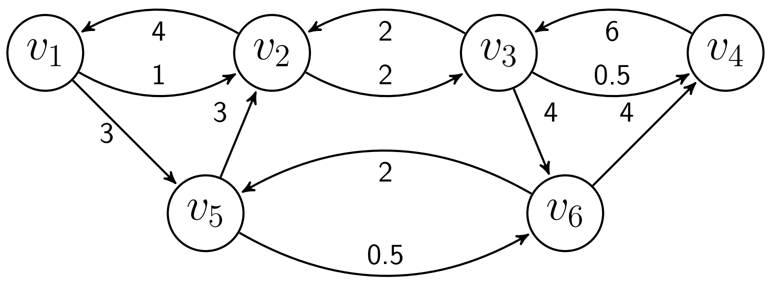
IEOR 151 – Homework 2
Due Friday, October 17, 2014 in class

1. Consider the following graph representation of a kidney exchange. Find the social welfare maximizing exchange under the constraint that all cycles can have length less than or equal to $L = 3$. (5 points)



2. Match the applicants to the residency programs, and show intermediate steps of the algorithm. (5 points)

   For this problem, suppose the applicants' preferences are given by:

Bob	**Louise**	**Lisa**
1. City	1. City	1. City
2. General	2. General	2. General
3. Mercy	3. Mercy	3. Mercy

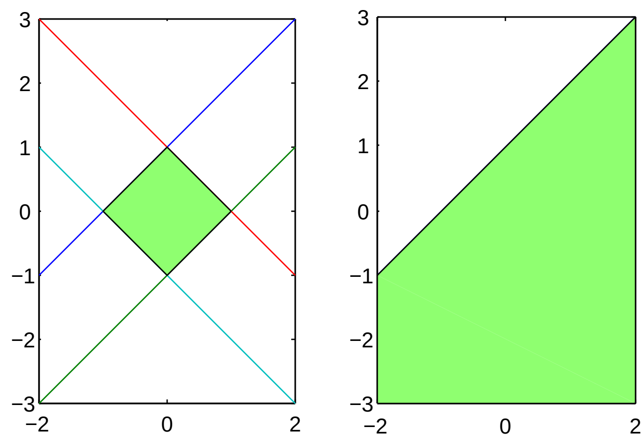
   Suppose that each residency program has only 1 open position, and that the preferences of the programs are given by

City	**General**	**Mercy**
1. Lisa	1. Lisa	1. Lisa
2. Louise	2. Bob	2. Louise
3. Bob	3. Louise	3. Bob

3. Consider the following optimization problem denoted (P)

$$\min\ x_1 - x_2$$
$$\text{s.t. } |x_1| + |x_2| \leq 1.$$

(a) The constraint $|x_1|+|x_2|-1 \leq 0$ is not differentiable, and so we cannot use the KKT conditions to solve this problem. This can be resolved by rewriting this constraint so that it is differentiable. Note that the corresponding feasible set is indicated by the shaded green square on the left side of the image below. One way to represent this feasible set is as the intersection of four half-spaces. For instance, the half-space $-x_1 + x_2 - 1 \leq 0$ is shown on the right side of the image below.



Write the constraint $|x_1| + |x_2| - 1 \leq 0$ as the intersection of four half-spaces. (2 points)

(b) Another representation of the constraint $|x_1| + |x_2| - 1 \leq 0$ is as the projection of a set. For example, we can write $|x_1| \leq 1$ as the projection of a polytope by introducing an additional variable t_1:

$$-t_1 \leq x_1 \leq t_1$$
$$t_1 = 1$$

Write the constraint $|x_1|+|x_2|-1 \leq 0$ as the projection of a polytope by introducing two additional variables t_1, t_2. (2 points)

(c) Suppose we had a constraint $\sum_{i=1}^{n} |x_i| - 1 \leq 0$, which is not differentiable. Write this constraint as the intersection of half-spaces. How many variables do we need? How many constraints/inequalities are there? (4 points)

(d) Suppose we had a constraint $\sum_{i=1}^{n} |x_i| - 1 \leq 0$, which is not differentiable. Write this constraint as the projection of a polytope. How many variables do we need? How many constraints/inequalities are there? (4 points)

(e) Recall the original optimization problem (P). Reformulate the optimization problem (P) so that the constraints are differentiable, and then use the KKT conditions to solve the optimization problem. Hint: There are an infinite number of solutions to (P). (4 points)

4. Suppose a fast food restaurant would like to purchase veggie burger patties from a food distributor. The restaurant's utility for the patties is given by $S(q) = 500ln(1 + q)$. The fixed costs for the food distributor are $4,000$, and if the distributor is inefficient (efficient) then its marginal costs are 0.15 (0.10). Assume that the restaurant believes that there is a 30% chance that the food distributor is efficient.

   (a) What are the first-best production levels? (2 points)

   (b) What are the contracts to implement the first-best production levels? (2 points)

   (c) How much profit would the efficient distributor make if the restaurant offers a menu of contracts $\{(q_1^I, t_1^I), (q_1^E, t_1^E)\}$? (1 point)

   (d) What are the second-best production levels? (2 points)

   (e) What is the menu of contracts for the second-best production levels? (2 points)

   (f) What is the information rent of the efficient distributor for the menu of contracts for the second-best production levels? Is this higher or lower than the profit gained for the menu of contracts for the first-best production levels? (2 points)