IEOR 151 – Lecture 14
Vertex P-Center Problem

1 Problem Setup

Location planning involves specifying the physical position of facilities that provide demanded services. Examples of facilities include hospitals, restaurants, ambulances, retail and grocery stores, schools, and fire stations. There are a variety of different models to solve this problem, and we will next consider another type of a discrete location model. These are models in which the facilities and demands are in discrete positions.

The vertex p-center problem is a specific type of a discrete location model. In this model, we wish to place p facilities to minimize the maximum distance between any demand node and the location in which a facility was placed. In this model, there are no capacity constraints at the facilities.

1.1 Mathematical Model

We represent the problem formulation using an undirected graph $G = (I, J, E)$, where the demand nodes are represented by a set of vertices $i \in I$, the possible locations of the facilities are given by another set of vertices $j \in J$, and edges $e_{i,j} \in E$ only exist between vertices from $i \in I$ to those in $j \in J$. Furthermore, we assign positive weights to the edges $d_{i,j} \geq 0$, which represents a distance between vertices i and j. Note that it is possible to have zero distance between a demand node and a possible facility location. In contrast to the p-median problem, we do *not* assign demand weights to each demand node. In this problem, we would like to place p facilities to minimize the maximum distance between any demand node and its servicing facility.

1.2 ILP Formulation

Because there are no capacity constraints at the facilities, we can assume that each demand node is serviced by a single facility. To formulate an integer linear program (ILP) to solve this problem, we define a decision variable

$$Y_{i,j} = \begin{cases} 1, \text{if demand node } i \in I \text{ assigned to facility located at } j \in J \\ 0, \text{otherwise} \end{cases}$$

that describes which demand nodes are serviced by which facility location. We need to define another decision variable

$$X_j = \begin{cases} 1, \text{if facility located at } j \in J \\ 0, \text{otherwise} \end{cases}$$

that describes the locations at which a facility is placed. In contrast to the p-median problem, we must define an additional decision variable Q that represents the maximum distance between any demand node and its servicing facility. Given these decision variables, we can now formulate the vertex p-center problem as the following ILP

$$\begin{aligned}
&\min Q \\
\text{s.t. } &\sum_{j\in J} Y_{i,j} = 1, \forall i \in I \\
&Y_{i,j} - X_j \leq 0, \forall i \in I, j \in J \\
&\sum_{j\in J} X_j = p \\
&\sum_{j\in J} d_{i,j} Y_{i,j} - Q \leq 0, \forall i \in I \\
&X_j \in \{0,1\}, \forall j \in J \\
&Y_{i,j} \in \{0,1\}, \forall i \in I, j \in J.
\end{aligned}$$

Each of these terms has associated intuition. The objective Q is stating that we wish to minimize the maximum distance between any demand node and its servicing facility. The first constraint $\sum_{j\in J} Y_{i,j} = 1$ implies that a demand node i can only be serviced by one facility. The constraint $Y_{i,j} - X_j \leq 0$ says that demand node i can be serviced by a facility at j only if there is a facility at j, because if $X_j = 0$ then we must have that $Y_{i,j} = 0$. The constraint $\sum_{j\in J} X_j = p$ means that we must place exactly p facilities. Next, the constraints that $X_j \in \{0,1\}$ and $Y_{i,j} \in \{0,1\}$ force the decision variables to be binary.

The constraint $\sum_{j\in J} d_{i,j} Y_{i,j} - Q \leq 0$ does not appear in the p-median problem. The term $\sum_{j\in J} d_{i,j} Y_{i,j}$ gives the distance between the i-th demand node and its servicing facility, because the constraint $\sum_{j\in J} Y_{i,j} = 1$ means that only one of the $Y_{i,j}$ variables can be nonzero. Thus, the constraint $\sum_{j\in J} d_{i,j} Y_{i,j} - Q \leq 0$ means that Q must be greater than the distance between the i-th demand node and its servicing facility. But because we have this constraint for each of the $i \in I$ demand nodes, this means that Q must be greater than the distance of any demand node and its servicing facility.

1.3 Heuristic Algorithm

The ILP can be difficult to solve, and so we discuss an heuristic algorithm to solve this problem. The heuristic is a greedy algorithm that randomly places the first facility and then places the remaining facilities in a greedy manner.

1. Randomly place the first facility, and put it into the set S;
2. For $i = 2, \ldots, p$
   (a) For $j = 1, \ldots, p - i$

  i. Sort the values $\min_{m \in S} d_{k,m}$ into descending order;
  ii. Select the demand node $k \in I$ such that $\min_{m \in S} d_{k,m}$ is the j-th value in descending order;
  iii. Find the potential facility location $n \in J$ such that $d_{k,n}$ is minimal;
  iv. If $n \notin S$, then exit the loop;

(b) Place a facility at $n \in J$, and add it to the set S;