

IEOR151 Homework 8 Solution

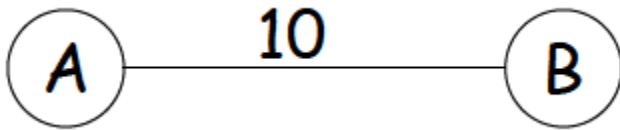
Fall 2013

Problem 1:

One of the important issues in facility location modeling is the following. If you are given a problem on a network composed of nodes and links, when do you get the same solution (in terms of the objective function value) if you restrict the facility location to be on the nodes as opposed to the case in which you allow facilities to be on the nodes and the links. Clearly life is much easier if you know that restricting the candidate sites to be the set of nodes does not hurt you as you have a finite set of candidate locations as opposed to the infinite set of candidate locations you would have if you could locate on the nodes and links. **(If you think they are different, give an example; If not, please explain thoroughly.)**

- a) For the set covering problem, do you get the same results if (i) you can locate only on the nodes and (ii) you can locate on the nodes and links? Prove that you either do get the same results or show by example that the results will be different depending on where you are allowed to locate.

Consider the following network with a coverage distance of 8. It is clear that if we are constrained to locate on the nodes only, we will need two facilities, but if we can locate anywhere near the middle of the link we will need only 1 facility. Thus, for the set covering model, it is better to be able to locate on the nodes AND links rather than to be constrained to locate on the nodes only.



- b) Repeat question (a) for the maximal covering problem.

Again, the network above shows that if $P=1$, if we were constrained to locate on the nodes only, we would locate at either A or B depending on which had more demand. The other node would not be covered. However, if we could locate on the link at any point that is more than 2 units from either node, we could cover both demand nodes (and all of the network's demand) with only one site. Thus, for the maximal covering problem it is better to be able to locate on the nodes and the links than to be constrained to locating only on the nodes.

- c) Repeat question (a) for the P-center problem.

Again, if we consider the network above, if we have to locate on the nodes only, the maximum distance (e.g., the P-center objective function value) is 10, whereas if we can locate in the middle of the link, the objective function value is 5. Thus, for the P-center problem it is better to be able to locate on the nodes and the links than to be constrained to locating only on the nodes.

- d) Repeat question (a) for the P-median problem.

In this case, we can show that there exists at least one optimal solution that consists of locating only on the nodes of the network. Suppose we have an allegedly optimal solution that consists of locating some facility on a link between nodes A and B, for example. Let us denote by H_A the demand that is served at that facility and that enters the facility via A. Similarly, define H_B as the demand that is served by the facility and that enters the facility via node B. Assume, without loss

of generality that $H_A > H_B$. Then we can move the facility toward A and we will not degrade the objective function since the demand weighted total distance for the demands served by the facility through A will go down by at least as much as the demand weighted total distance of the demands served by the facility through B goes up. (If $H_A > H_B$, such a move will actually improve the objective function.) We can continue moving the facility in this way until it gets to node A. Thus, we can always find a solution that consists of locating only on the nodes.

Problem 2:

For every finite undirected graph $G(N,E)$, please prove the following equation:

$$\sum_{i \in N} \text{deg}(i) = 2|E|$$

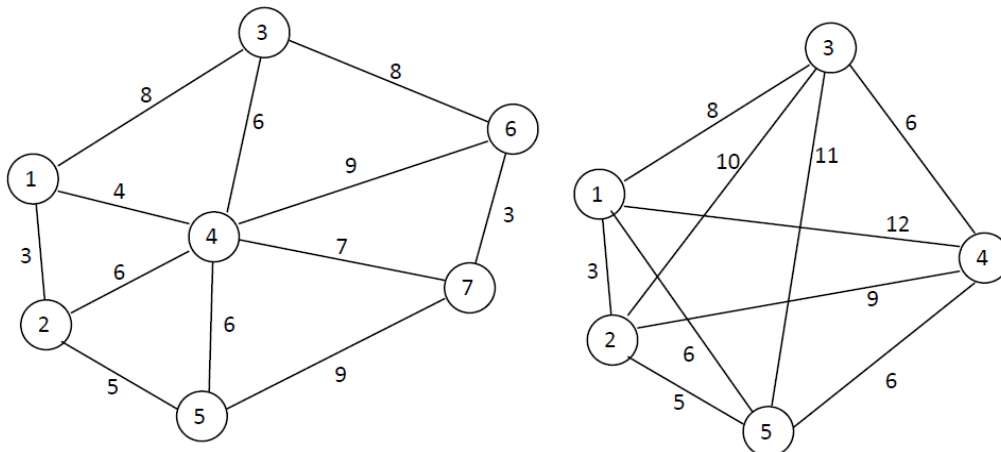
where $\text{deg}(i)$ is the degree of node $i \in N$ and $|E|$ is the number of edges in the graph.

Euler's proof of the degree sum formula uses the technique of double counting: he counts the number of incident pairs (v,e) where e is an edge and vertex v is one of its endpoints, in two different ways. Vertex v belongs to $\text{deg}(v)$ pairs, where $\text{deg}(v)$ (the degree of v) is the number of edges incident to it. Therefore the number of incident pairs is the sum of the degrees. However, each edge in the graph belongs to exactly two incident pairs, one for each of its endpoints; therefore, the number of incident pairs is $2|E|$. Since these two formulas count the same set of objects, they must have equal values.

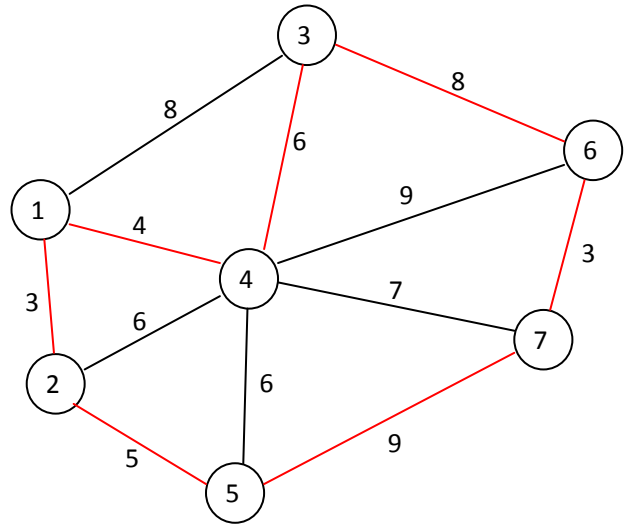
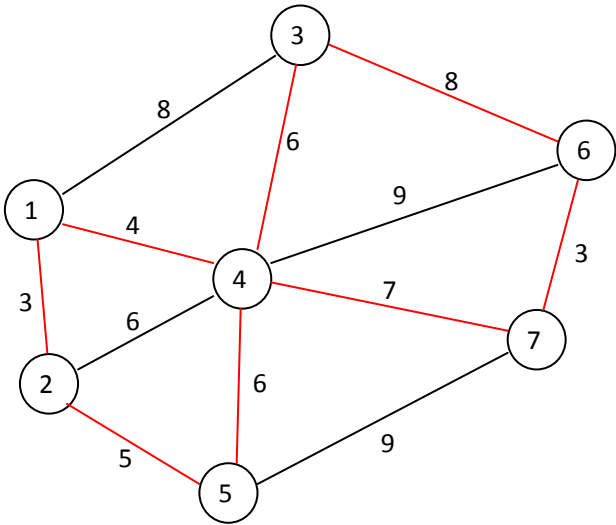
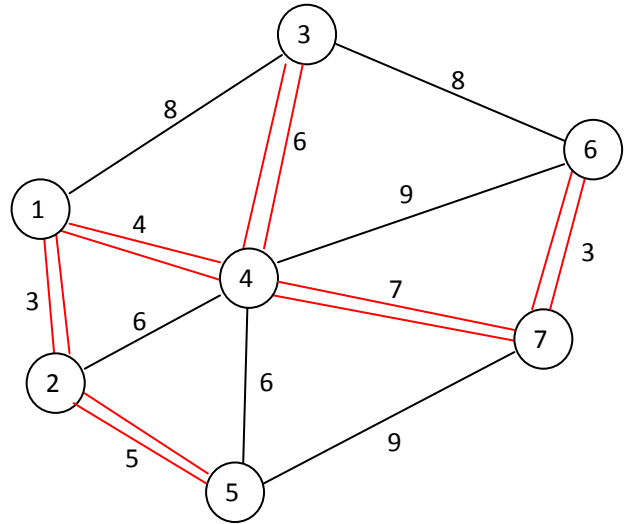
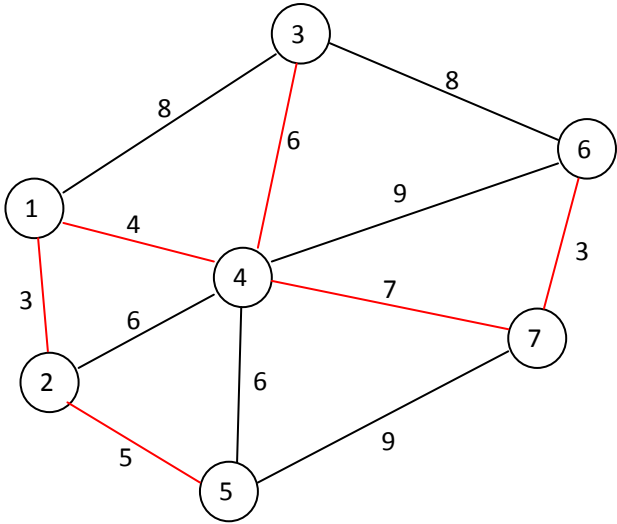
In a sum of integers, the parity of the sum is not affected by the even terms in the sum; the overall sum is even when there is an even number of odd terms, and odd when there is an odd number of odd terms. Since one side of the degree sum formula is the even number $2|E|$, the sum on the other side must have an even number of odd terms; that is, there must be an even number of odd-degree vertices.

Problem 3:

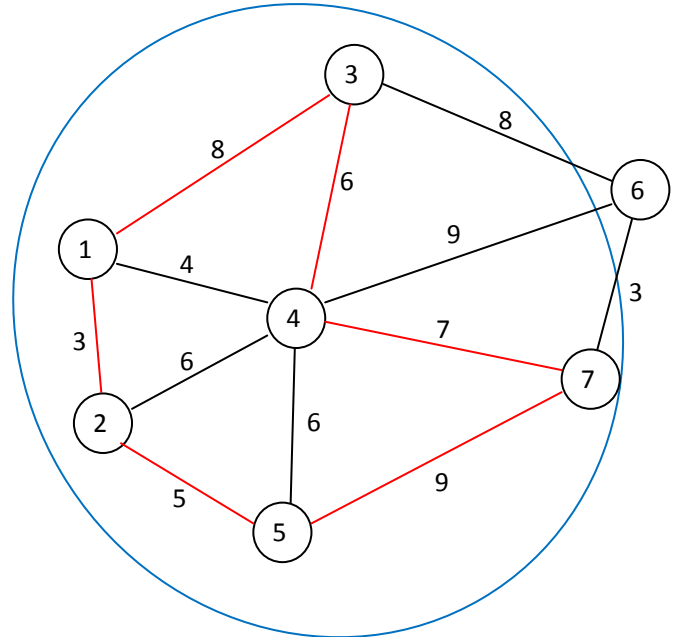
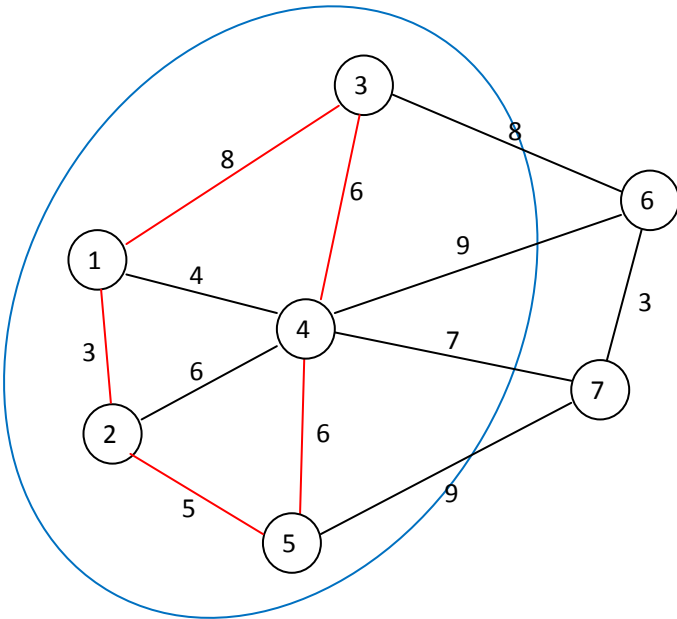
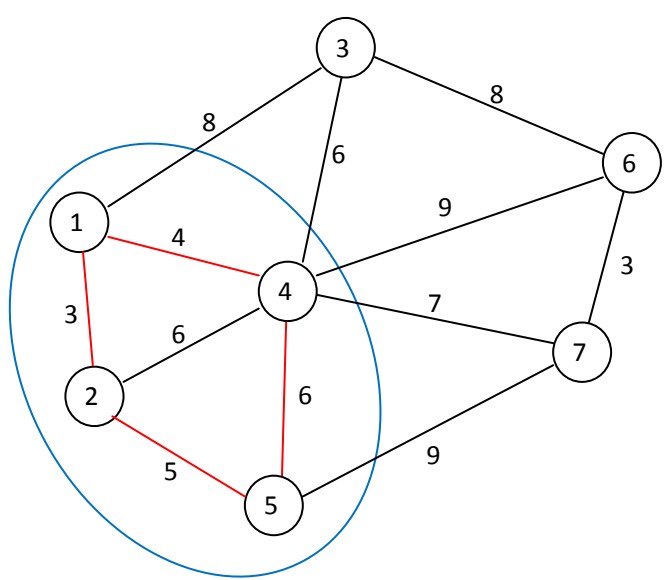
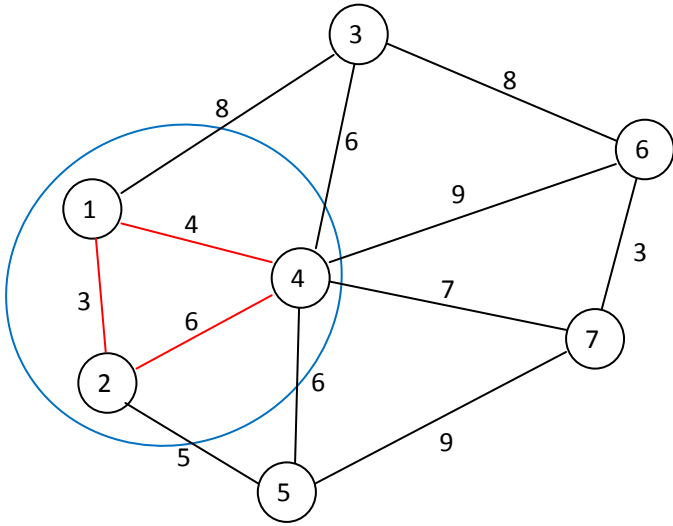
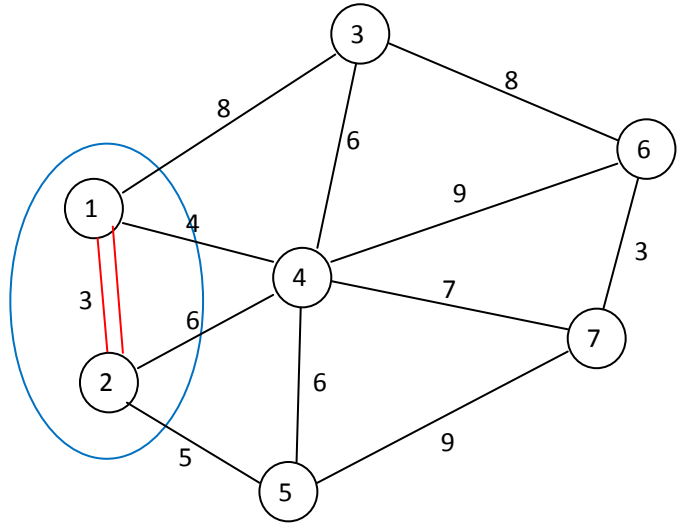
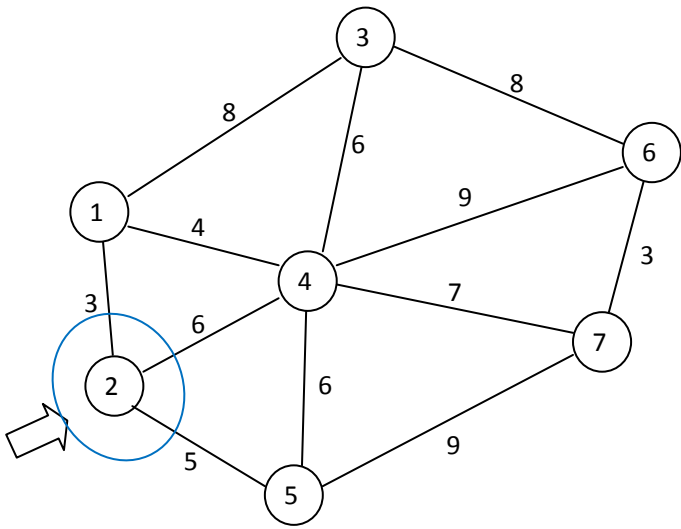
Find the traveling salesman tours in the left graph using the minimal spanning tree based heuristic and the right graph using the nearest insertion heuristic, respectively. Report the total distances and sequences of nodes.

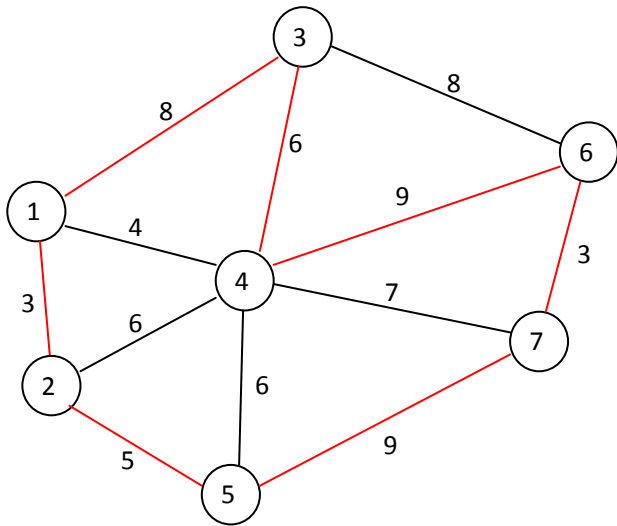


1. MST based heuristic

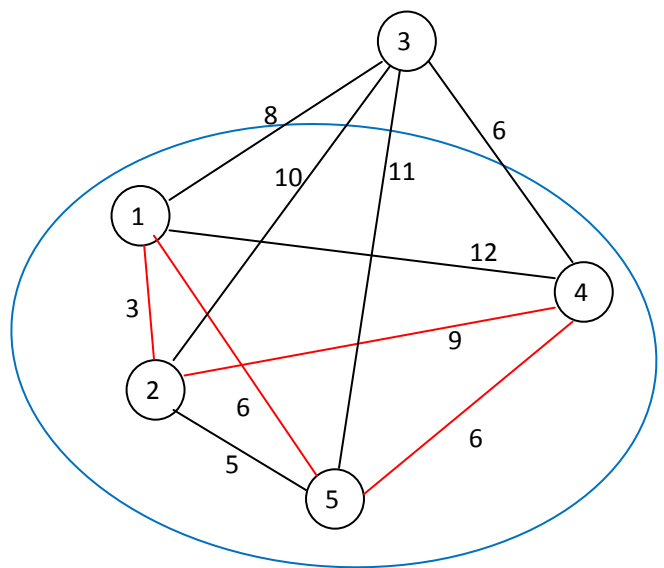
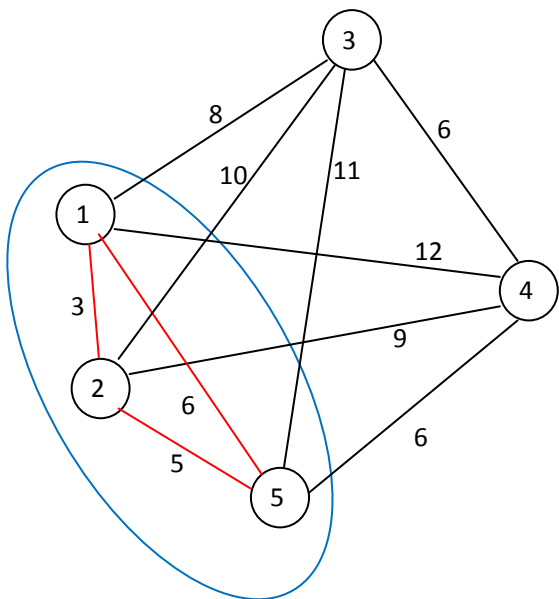
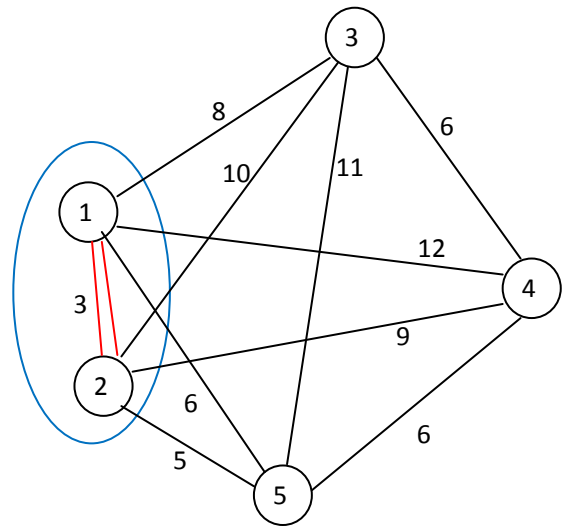
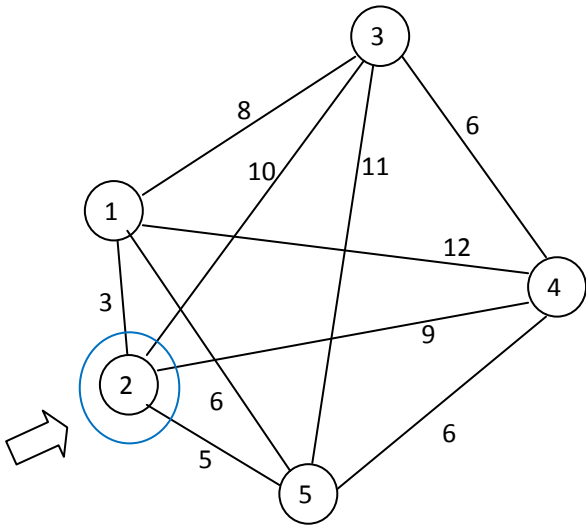


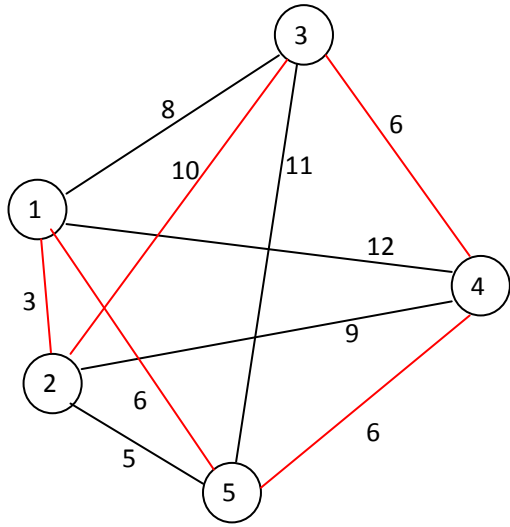
Nearest Insertion I



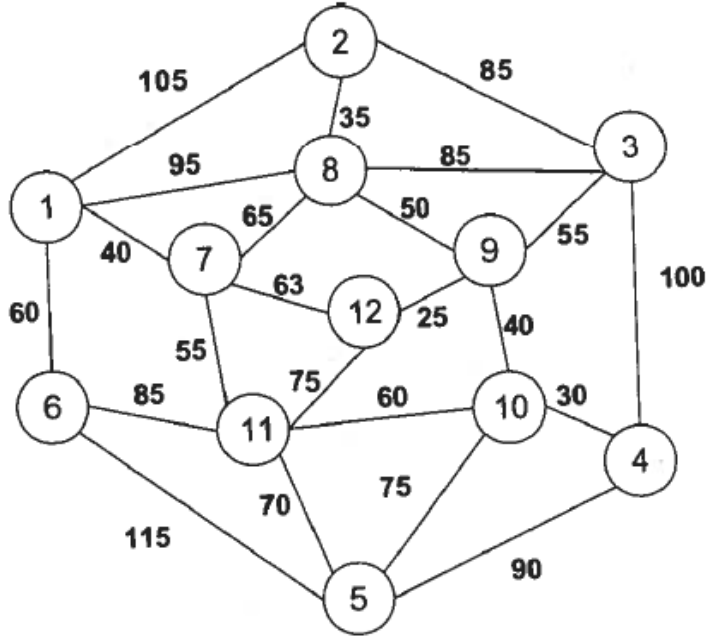


Nearest Insertion II





Problem 4:



Solve the Chinese Postman Problem.

- What are the odd-degree nodes?
2, 4, 6, 8, 11, 12
- What is the optimal pairing of the odd-degree nodes?
2-8, 6-11, 4-12
- How much additional distance is added due to the pairing of the odd degree nodes?
215
- What is the total length of the original network (without the total distance of the optimal pairing of the odd-degree nodes)?
1558
- What percentage of the total length of the original network is the total distance of the optimal pairing of the odd-degree nodes?
13.8%
- Find an Euler tour on the original network plus the links that are added as a result of the optimal pairing of the odd-degree nodes. List the nodes to be visited in order beginning with node 1.
1-2-8-1-7-8-2-3-8-9-3-4-10-4-5-6-11-5-10-9-12-7-11-12-9-10-11-6-1
Note that this tour is NOT unique and there are many other Euler tours possible. The key is that you have to traverse each link exactly once except for the red links (2-8; 4-10; 10-9; 9-12; and 6-11) which have to be traversed twice.

Problem 5

Consider the data shown below for a vehicle routing problem. The depot is at node 0. Each vehicle has a capacity of 350 units. The maximum distance that a vehicle can travel is 50 units. Compute the distances using the Euclidean distance formula. That is, the distance, d_{jk} between points j and k , with coordinates

$$(x_j, y_j) \text{ and } (x_k, y_k), \text{ respectively, is } d_{jk} = \sqrt{(x_j - x_k)^2 + (y_j - y_k)^2}.$$

Node	X-coordinate	Y-coordinate	Demand	Angle	Radius
0	0	0	0		
1	5	7	65	54.46	8.60
2	3	5	95	59.04	5.83
3	6	-6	100	315.00	8.49
4	-5	-1	85	191.31	5.10
5	-3	-9	60	251.57	9.49
6	-7	-6	110	220.60	9.22
7	3	-8	100	290.56	8.54
8	7	2	85	15.95	7.28
9	-3	4	60	126.87	5.00
10	-4	8	75	116.57	8.94

Use the savings algorithm to solve the problem.

First, draw the demand nodes on graph with XY coordinates.

It is better to use Excel to compute the distances and savings.

Distance	0	1	2	3	4	5	6	7	8	9	10
0	0.00	8.60	5.83	8.49	5.10	9.49	9.22	8.54	7.28	5.00	8.94
1	8.60	0.00	2.83	13.04	12.81	17.89	17.69	15.13	5.39	8.54	9.06
2	5.83	2.83	0.00	11.40	10.00	15.23	14.87	13.00	5.00	6.08	7.62
3	8.49	13.04	11.40	0.00	12.08	9.49	13.00	3.61	8.06	13.45	17.20
4	5.10	12.81	10.00	12.08	0.00	8.25	5.39	10.63	12.37	5.39	9.06
5	9.49	17.89	15.23	9.49	8.25	0.00	5.00	6.08	14.87	13.00	17.03
6	9.22	17.69	14.87	13.00	5.39	5.00	0.00	10.20	16.12	10.77	14.32
7	8.54	15.13	13.00	3.61	10.63	6.08	10.20	0.00	10.77	13.42	17.46
8	7.28	5.39	5.00	8.06	12.37	14.87	16.12	10.77	0.00	10.20	12.53
9	5.00	8.54	6.08	13.45	5.39	13.00	10.77	13.42	10.20	0.00	4.12
10	8.94	9.06	7.62	17.20	9.06	17.03	14.32	17.46	12.53	4.12	0.00

Sort savings in descending order

Savings	Pair	Savings	Pair	Savings	Pair	Savings	Pair	Savings
13.71	2,8	8.11	4,9	4.71	1,7	2.01	3,10	0.22
13.42	3,8	7.70	3,6	4.70	5,8	1.90	1,5	0.20
11.95	6,7	7.57	1,3	4.05	3,4	1.50	2,6	0.18
11.60	2,10	7.16	6,10	3.85	5,9	1.49	1,6	0.13
10.50	4,5	6.34	8,10	3.69	5,10	1.40	7,9	0.13
9.82	1,9	5.06	6,9	3.45	2,7	1.37	2,5	0.09
8.93	7,8	5.05	4,7	3.01	2,4	0.93	3,9	0.03
8.49	4,10	4.99	2,3	2.91	1,4	0.90	7,10	0.02
8.49	2,9	4.75	8,9	2.08	6,8	0.38	4,8	0.01

- a) How many routes do you use?
3
- b) What is the total route length?
84.78
- c) List the nodes visited on each route, the demand associated with that route, and the route length.
Route 0-2-1-8-0 with demand 245 and distance 21.33;
Route 0-7-3-10-9-0 with demand 335 and distance 38.47;
Route 0-4-6-5-0 with demand 255 and distance 24.98.

Problem 6

Exercise 1 on page 504 in textbook chapter 9

Arrival rate $\lambda = 54/\text{hr} = 0.9/\text{min}$

Service rate $\mu = 1/3$ per min.

(a) Use M/M/3 queue with one line formula:

$$W_q \approx \frac{\left(\frac{\lambda}{s\mu}\right)\sqrt{2(s+1)-1}}{s\left(1-\frac{\lambda}{s\mu}\right)} \cdot \frac{1}{\mu} = 8.23 \text{ mins}$$

The waiting time in queue is 8.23 minutes.

(b) Use M/M/3 queue with 3 lines formula:

$$W = \frac{1}{\mu - \lambda/3} = 30 \text{ mins}$$

$$W_q = W - W_s = 27 \text{ mins}$$

Problem 7

Exercise 11 on page 455 in textbook chapter 8

Let $t=1, \dots, 15$, where $t=1$ means Oct in previous year and so on.

$A_t = \#$ of employees in month t

$T_t = \#$ of trainees hired in month t

$D_t = \#$ of needed employees in month t .

Formulation.

$$\min \quad 2500 \sum_{t=1}^{12} A_t + 1500 \sum_{t=1}^{12} T_t + 15000 \sum_{t=1}^{12} X_t$$

$$\text{s.t.} \quad A_t = 0.9A_{t-1} + T_{t-3}, \quad t=4, \dots, 15.$$

$$T_{13} = T_{14} = T_{15} = 0$$

$$A_3 = 100;$$

$$A_t \geq D_t,$$

$$1000 X_1 \geq T_1$$

$$1000 X_2 \geq T_1 + T_2$$

$$1000 X_t \geq T_t + T_{t-1} + T_{t-2}, \quad t=3, \dots, 12.$$

$$X_t \in \{0, 1\}, \quad t=1, \dots, 12$$

$$A_t \geq 0, \quad t=1, \dots, 15$$

$$T_t \geq 0, \quad t=1, \dots, 15.$$