IEOR 151 – Homework 5
Due Friday, October 11, 2013 in class

1. Consider the following optimization problem

$$\begin{aligned} &\min x \\ &\text{s.t. } x \in \mathbb{R} \\ &\quad -x^3 \leq 0. \end{aligned}$$

(a) If $x^* \in \{x \in \mathbb{R} : -x^3 \leq 0\}$ is the global minimizer of this optimization problem, then what are Fritz John conditions? (2 points)

The Fritz John conditions are that there exists λ_0, λ_1 such that

$$\begin{aligned} &\lambda_0 - 3\lambda_1 x^{*2} = 0 \\ &\lambda_0, \lambda_1 \geq 0 \\ &-\lambda_1 x^{*3} = 0 \\ &(\lambda_0, \lambda_1) \neq 0. \end{aligned}$$

(b) Determine all possible combinations of $(\lambda_0, \lambda_1, x^*)$ for which the Fritz John conditions hold. (3 points)

We begin with the complimentary slackness condition: $-\lambda_1 x^{*3} = 0$. This means that either (a) $\lambda_1 = 0$ and $x^* : -x^{*3} \leq 0$, or (b) $\lambda_1 > 0$ and $x^* = 0$. For case (a), the stationarity condition: $\lambda_0 - 3\lambda_1 x^{*2} = 0$ implies that $\lambda_0 = 0$; however, this violates $(\lambda_0, \lambda_1) \neq 0$ and hence cannot be a solution. For case (b), the stationarity condition: $\lambda_0 - 3\lambda_1 x^{*2} = 0$ implies that $\lambda_0 = 0$. Summarizing, the only set of solutions are

$$\begin{aligned} &\lambda_0 = 0 \\ &\lambda_1 : \lambda_1 > 0 \\ &x^* = 0. \end{aligned}$$

(c) Show that the LICQ does not hold at all feasible points? (1 point)

If $g(x) = -x^3 \leq 0$, then its gradient is given by $\nabla g(x) = -3x^2$. Thus, the LICQ does not hold at $x = 0$ because $\nabla g(0) = 0$, which is linearly dependent since it is zero.

(d) If the global minimizer x^* satisfies the KKT conditions, then write down the KKT conditions. If the global minimizer x^* does not satisfy the KKT conditions, explain why. (2 points)

The global minimizer x^* does not satisfy the KKT conditions because the LICQ does not hold.

(e) Rewrite the optimization problem so that the LICQ holds for all feasible points. (2 points).

The constraint $-x^3 \leq 0$ can be simplified to $-x \leq 0$. So if we rewrite the optimization problem as

$$\begin{aligned}
&\min x \\
&\text{s.t. } x \in \mathbb{R} \\
&\qquad - x \leq 0,
\end{aligned}$$

then the LICQ holds because for $g(x) = -x$, the gradient $\nabla g(x) = 1$ is linearly independent since it is nonzero.

(f) What are the KKT conditions for the rewritten optimization problem where the LICQ holds? (2 points)

The KKT conditions are that there exists λ_1 such that

$$\begin{aligned}
&1 - \lambda_1 = 0 \\
&\lambda_1 \geq 0 \\
&- \lambda_1 x^* = 0 \\
&(\lambda_0, \lambda_1) \neq 0.
\end{aligned}$$

(g) Determine all possible combinations of (λ_1, x^*) for which the KKT conditions hold. Based on these combinations, compute the global minimizer of the optimization problem? (3 points)

From the stationarity condition: $1 - \lambda_1 = 0$, we have that $\lambda_1 = 1$. Combining this with the complimentary slackness condition: $-\lambda_1 x^* = 0$ gives that $x^* = 0$. This is the only possible solution of the KKT conditions, and so the local minimizer must also be the global minimizer. Summarizing, the global minimizer is $x^* = 0$.

2. Consider the following parametric optimization problem

$$\begin{aligned}
V(\theta) = &\min_x \theta x \\
&\text{s.t. } x \in [-1, 1]
\end{aligned}$$

(a) Is the objective jointly continuous in (x, θ)? Is the constraint set $[1, 1]$ continuous in θ? Hint: You do not have to do any calculations. (2 points)

By inspection, the objective is jointly continuous in (x, θ) because it is simply the product of x and θ. And because the constraint set is constant over different values of θ, it is also continuous in θ.

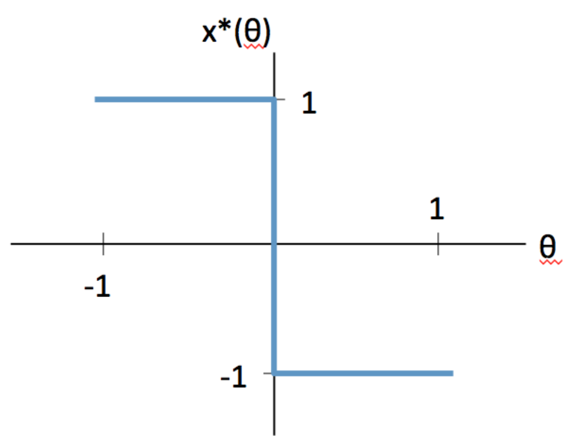
(b) Compute the minimizer $x^*(\theta)$ for $\theta \in [-1, 1]$? (3 points)

We break the problem into three cases. In the first case $\theta > 0$, and we are solving $\min\{|\theta|x : x \in [-1, 1]\}$. Clearly, the minimizer is $x^* = -1$. In the second case $\theta < 0$, and we are solving $\min\{-|\theta|x : x \in [-1, 1]\}$. Clearly, the minimizer is $x^* = 1$. The last case is when $\theta = 0$. Here, we are solving $\min\{0 : x \in [-1, 1]\}$, and so the minimizer is any value $x^* \in [-1, 1]$. Summarizing, the minimizers are

$$x^*(\theta) \in \begin{cases} 1, & \text{if } \theta < 0 \\ [-1, 1], & \text{if } \theta = 0 \\ -1, & \text{if } \theta > 0 \end{cases}$$

(c) Plot $x^*(\theta)$. Is it upper hemi-continuous? (2 point)

The plot is below, and yes – $x^*(\theta)$ is upper hemi-continuous because of the Closed Graph Theorem.

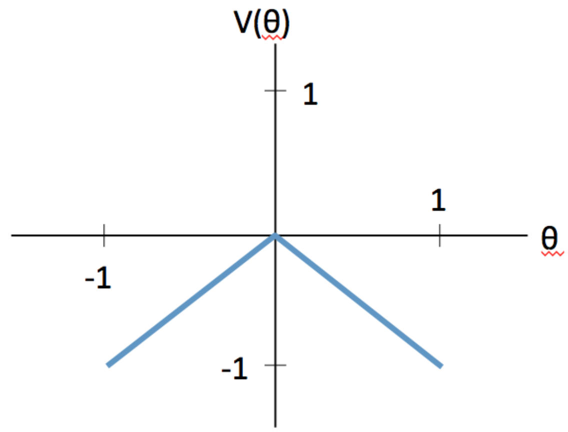


(d) Compute the value function $V(\theta)$ for $\theta \in [-1, 1]$? (1 point)

Substituting $x^*(\theta)$ into the objective gives $V(\theta) = -|\theta|$ for $\theta \in [-1, 1]$.

(e) Plot $V(\theta)$. Is it continuous? (2 point)

The plot is below, and yes – $V(\theta)$ is continuous by observation.



(f) Do these results agree with the Berge Maximum Theorem? Is this expected? (1 point)

Yes, the results agree with the Berge Maximum Theorem. This is expected because the objective is jointly continuous in (x, θ) and the constraints are continuous in θ.