IEOR 151 – Homework 1
Due Wednesday, September 11, 2013 in class

1. Two players A and B play a game in which they take turns flipping a coin. A starts the game, and B plays second. The first player to flip a heads wins.

   (a) Compute the probability that A wins if the coin is fair – meaning the probability of heads is $1/2$. (2 points)

   We use a_i and b_i to denote the i-th flip of players A and B, respectively. Let h and t denote heads and tails. Note that

   $$\begin{aligned} \mathbb{P}(A \text{ wins}) &= \mathbb{P}(a_1 = h) + \mathbb{P}(a_1 = t \text{ and } b_1 = t \text{ and } A \text{ wins}) \\ &= \mathbb{P}(a_1 = h) + \mathbb{P}[A \text{ wins}|a_1 = t \text{ and } b_1 = t] \cdot \mathbb{P}(a_1 = t \text{ and } b_1 = t) \\ &= \mathbb{P}(a_1 = h) + \mathbb{P}(A \text{ wins}) \cdot \mathbb{P}(a_1 = t) \cdot \mathbb{P}(b_1 = t) \\ &= 1/2 + 1/4 \cdot \mathbb{P}(A \text{ wins}). \end{aligned}$$

   Solving for $\mathbb{P}(A \text{ wins})$ gives the answer: $\mathbb{P}(A \text{ wins}) = 2/3$.

   (b) Compute the probability that A wins if the coin is biased, and the probability of heads is p such that $p > 0$. (1 point)

   From the derivation in the previous part of the question, it holds that

   $$\mathbb{P}(A \text{ wins}) = \mathbb{P}(a_1 = h) + \mathbb{P}(A \text{ wins}) \cdot \mathbb{P}(a_1 = t) \cdot \mathbb{P}(b_1 = t) = p + (1 - p)^2 \cdot \mathbb{P}(A \text{ wins}).$$

   Solving this gives $\mathbb{P}(A \text{ wins}) = p/(1 - (1 - p)^2) = 1/(2 - p)$.

   (c) For what values of p does A have a better chance of winning than B? (1 point)

   A has a better chance of winning if $\mathbb{P}(A \text{ wins}) > 1/2$. This occurs when

   $$\begin{aligned} 1/(2 - p) > 1/2 &\Leftrightarrow 2 > 2 - p \\ &\Leftrightarrow p > 0. \end{aligned}$$

   Thus, A always has a better chance of winning if $p > 0$.

2. Consider a disease that is found in 0.1% of the population, with a test for this disease that gives a correct diagnosis 90% of the time. Suppose a person is randomly chosen from the population and given a first test. If the first test is positive, then the person is given a second test.

   (a) Compute the probability that the first test is positive given that the person is infected with the disease. (1 point)

      Direct computation gives $\mathbb{P}[\text{One Positive Test}|\text{Infected}] = 0.9$.

   (b) Compute the probability that both tests are positive given that the person is infected with the disease. (1 point)

      Direct computation gives $\mathbb{P}[\text{Two Positive Tests}|\text{Infected}] = 0.9^2 = 0.81$.

   (c) Compute the probability that the person is infected given that the first test is positive. (2 points)

      First note that

      $$\mathbb{P}(\text{One Positive Test}) = \mathbb{P}[\text{One Positive Test}|\text{Infected}] \cdot \mathbb{P}(\text{Infected}) +$$
      $$\mathbb{P}[\text{One Positive Test}|\text{Not Infected}] \cdot \mathbb{P}(\text{Not Infected})$$
      $$= 0.9 \cdot 0.001 + 0.1 \cdot 0.999.$$

      Using Bayes' rule gives

      $$\mathbb{P}[\text{Infected}|\text{One Positive Test}] = \frac{\mathbb{P}[\text{One Positive Test}|\text{Infected}] \cdot \mathbb{P}(\text{Infected})}{\mathbb{P}(\text{One Positive Test})}$$
      $$= \frac{0.9 \cdot 0.001}{0.9 \cdot 0.001 + 0.1 \cdot 0.999} \approx 0.0089.$$

   (d) Compute the probability that the person is infected given that both tests are positive. (2 points)

      A similar computation gives $\mathbb{P}[\text{Infected}|\text{Two Positive Tests}] = 0.075$.

3. Consider an airplane with maximum seating capacity of 100 passengers, and assume that the airline company sells 110 tickets because each passenger has a 10% probability (independently of other passengers) of not showing up for the flight. What is the probability that there are enough seats for all passengers that show up for the flight? Hint: It is not necessary to mathematically simplify the answer. (2 points)

There are enough seats if the number of passengers that show up is 100 or less (or the number of passengers that do not show up is 10 or more), meaning

$$\mathbb{P}(\text{Enough Seats}) = \sum_{k=0}^{100} 0.1^{110-k} \cdot 0.9^k = \sum_{k=10}^{110} 0.1^k \cdot 0.9^{110-k}.$$

Another characterization is through negation of events, specifically

$$\mathbb{P}(\text{Enough Seats}) = 1 - \sum_{k=101}^{110} 0.1^{110-k} \cdot 0.9^k = 1 - \sum_{k=0}^{9} 0.1^k \cdot 0.9^{110-k}.$$

4. Suppose $g(u)$ is twice continuously differentiable at $u = \theta$, $g'(\theta) = 0$, and $g''(\theta) \neq 0$. If $\sqrt{n}[X_n - \theta] \overset{d}{\to} \sigma\mathcal{N}(0,1)$, then show that $n[g(X_n) - g(\theta)] \overset{d}{\to} \sigma^2 \frac{g''(\theta)}{2} \chi_1^2$, where χ_1^2 is a chi-squared random variable with 1 degree of freedom. Hint 1: If Z is distributed according to $\mathcal{N}(0,1)$, then Z^2 is distributed according to χ_1^2. Hint 2: Use the Lagrange form of Taylor's Theorem. (5 points)

Using Taylor's Theorem, there exists some $\xi(u,\theta)$ in between u and θ such that

$$g(u) = g(\theta) + g'(\theta)(u - \theta) + \frac{g''(\xi(u,\theta))}{2}(u-\theta)^2.$$

But $g'(\theta) = 0$ and so we can ignore the second term in the expression above. Now consider the expression:

$$n[g(X_n) - g(\theta)] = n\frac{g''(\xi(X_n,\theta))}{2}(X_n - \theta)^2.$$

We must have $\xi(X_n, \theta) \overset{p}{\to} \theta$ since (i) $\xi(u,\theta)$ is in between X_n and θ, and (ii) $X_n \overset{p}{\to} \theta$. Thus, the Continuous Mapping Theorem implies that $g''(\xi(X_n,\theta)) \overset{p}{\to} g''(\theta)$. Again using the Continuous Mapping Theorem gives that $n(X_n - \theta)^2 \overset{d}{\to} \sigma^2\chi_1^2$ because of Hint 1 and the assumption $\sqrt{n}[X_n - \theta] \overset{d}{\to} \sigma\mathcal{N}(0,1)$. Applying Slutsky's theorem to $n[g(X_n) - g(\theta)]$ and using the intermediate conclusions above gives the desired result.

5. Suppose $\sqrt{n}[\hat{p} - p] \overset{d}{\to} \sqrt{p(1-p)}\mathcal{N}(0,1)$. If $g(u) = 2\arcsin(\sqrt{u})$, then compute the limiting distribution of $\sqrt{n}[g(\hat{p}) - g(p)]$. Note: The function $g(u)$ in this case is what is known as a variance stabilizing transformation. (3 points)

Note that $g'(p) = 1/\sqrt{p(1-p)}$, and so using the Delta Method gives $\sqrt{n}[g(\hat{p}) - g(p)] \overset{d}{\to} \mathcal{N}(0,1)$.